Spoon Feeding Hyperbola

Simplified Knowledge Management Classes Bangalore

My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams (such as IMO [International Mathematics Olympiad], IPhO [International Physics Olympiad], IChO [International Chemistry Olympiad]), IGCSE (IB), CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25 th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education (HBCSE) Physics Olympics camp BARC Campus.

I am Life Member of …

- IAPT (Indian Association of Physics Teachers)

- IPA (Indian Physics Association)

- AMTI (Association of Mathematics Teachers of India)

- National Human Rights Association

- Men's Rights Movement (India and International)

- MGTOW Movement (India and International)

And also of

IACT (Indian Association of Chemistry Teachers)

The selection for National Camp (for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy) happens in the following steps ….

1) **NSEP** (National Standard Exam in Physics) and **NSEC** (National Standard Exam in Chemistry) held around 24 rth November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2) **INPhO** (Indian National Physics Olympiad) and **INChO** (Indian National Chemistry Olympiad). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3) The Top 35 students of each subject are invited at HBCSE (Homi Bhabha Center for Science Education) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of "Good Books". Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.

There are 3 kinds of Text Books

- The thin Books - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to "Cram" quickly and pass somehow find the thin books "good" as they have to read less !!

- The Thick Books - Most students do not like these, as they want to read as less as possible. Average students are "busy" with many other things and have no time to read all these.

- The Average sized Books - Good students do not get all details in any one book. Most bad students do not want to read books of "this much thickness" also !!

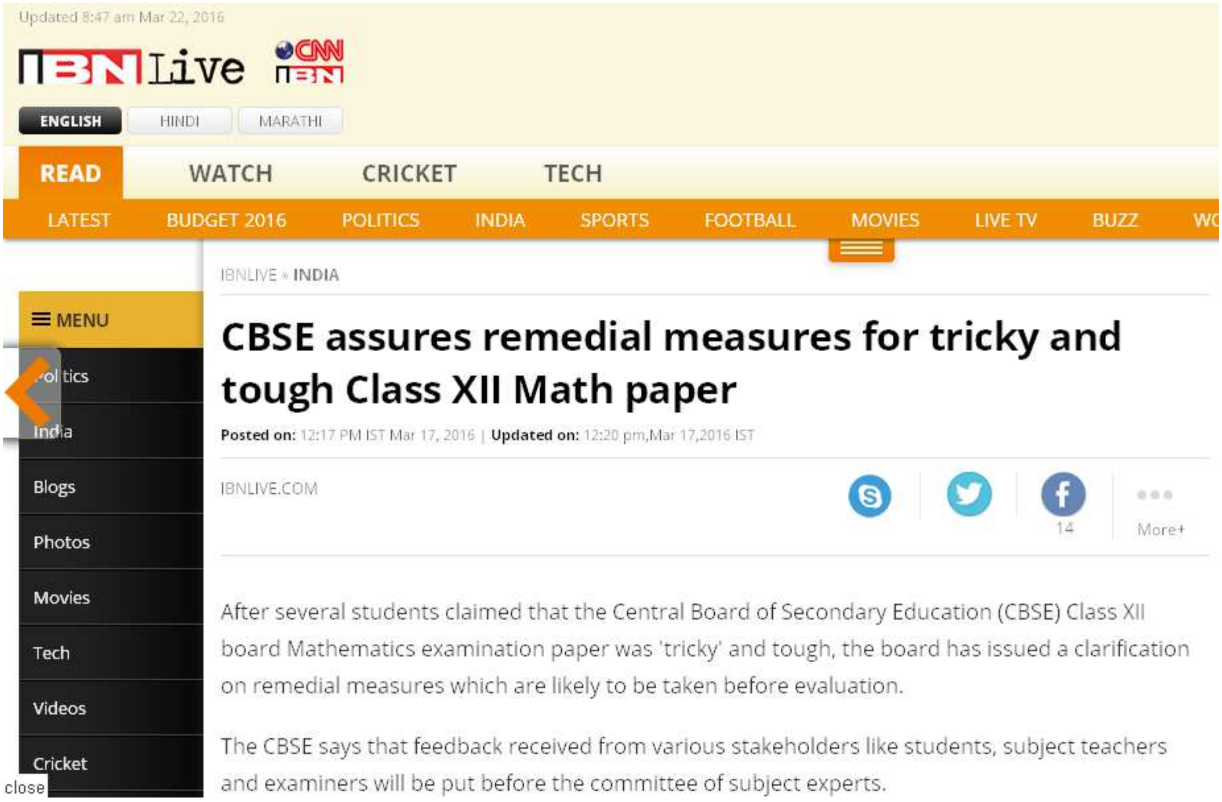
We know there can be no shoe that's fits in all.

Printed books are not e-Books! Can't be downloaded and kept in hard-disc for reading "later" ………

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good "Reference Material". I sincerely wish that all find this "very useful".

Students who do not practice lots of problems, do not do well. The rules of "doing well" had never changed …. Will never change !

After 2016 CBSE Mathematics exam, lots of students complained that the paper was tough!


Updated 8:47 am Mar 22, 2016

ENGLISH HINDI MARATHI

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CBSE assures remedial measures for tricky and tough Class XII Math paper

Posted on: 12:17 PM IST Mar 17, 2016 | **Updated on:** 12:20 pm,Mar 17,2016 IST

IBNLIVE.COM

After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.

On 21 st May 2016 the CBSE standard 12 result was declared. I loved the headline

INDIATODAY.IN NEW DELHI, MAY 21, 2016 | UPDATED 16:40 IST

CBSE Class 12 Results out: No leniency in Maths paper, high paper standard to be maintained in future

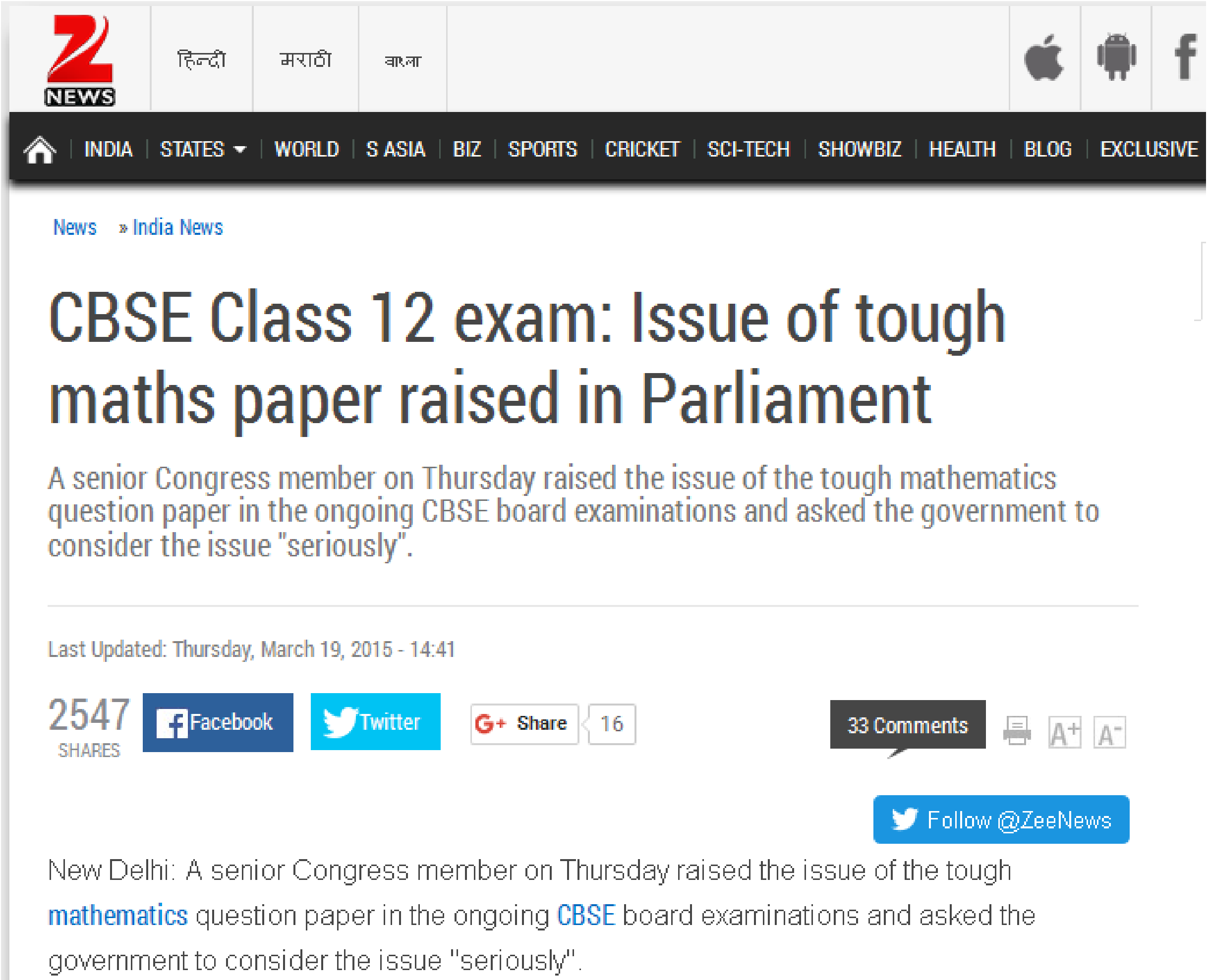
The CBSE Class 12 Mathematics board exam on March 14 reduced many students to tears as they found the paper quite lengthy and tough and many couldn't finish it on time. The results show an overall lowering of marks received in the Maths paper.

RELATED STORIES

- CBSE Board result 2016 declared! Thiruvanathpuram obtains the highest part percentage, check how your region scored
- Meet CBSE topper Sukriti Gupta: Check her percentage here!
- CBSE Class 12 Boards 2016: Results announced ahead of time!
- CBSE results declared at www.cbse.nic.in: Steps to check online
- Exclusive! CBSE declares Class 12 Results at www.cbseresults.nic.in and cbse.nic.in

The CBSE (Central Board of Secondary Education) Class 12 Board exam results have been announced today, i.e on May 21, around 10:30 am ahead of time. Students may check their scores at the official website, *www.cbseresults.nic.in*. (**Read: CBSE Class 12 Boards 2016: Results announced ahead of time! Check your score at cbseresults.nic.in**)

In 2015 also the same complain was there by many students



NEWS हिन्दी मराठी বাংলা

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News » India News

CBSE Class 12 exam: Issue of tough maths paper raised in Parliament

A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

Last Updated: Thursday, March 19, 2015 - 14:41

2547 SHARES Facebook Twitter Share 16 33 Comments

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New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

So we see that by raising frivolous requests, even upto parliament, actually does not help. Many times requests from several quarters have been put to CBSE, or Parliament etc for easy Math Paper. These kinds of requests actually can-not be entertained, never will be.

In March 2016, students of Karnataka PU-II also complained the same, regarding standard 12 (PU-II Mathematics Exam). Even though the Math Paper was identical to previous year, most students had not even solved the 2015 Question Paper.

Friday, March 25, 2016 - 13:28

The NEWS Minute

HOME NEWS ANDHRA KARNATAKA KERALA TAMIL NADU TELANGANA CULTURE MEDIA BLOG

Exams

Online petition for lenient evaluation of K'taka II PU math paper gets over 8000 supporters

The campaign, which was launched on Monday, has garnered over 8000 supporters

TNM Staff | Wednesday, March 16, 2016 - 09:32

Follow @thenewsminute

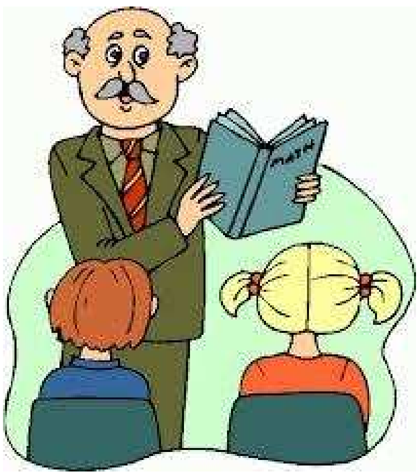
Share Tweet Email Share Reddit

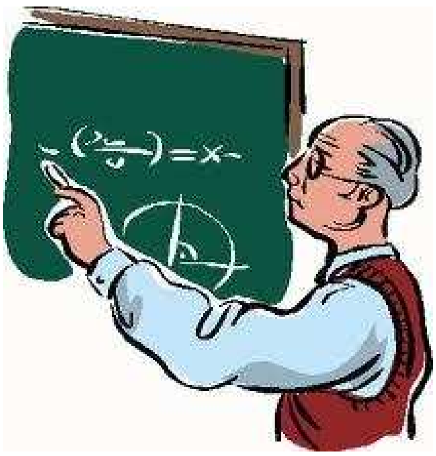
Following a "very tough" math paper that left many II PU students in tears, Saket Ravindran a student launched an online campaign demanding lenient evaluation.

These complains are not new. In fact since last 40 years, (since my childhood), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

No one can help those who are not studying, or practicing.





Learn more at http://skmclasses.weebly.com/iit-jee-home-tuitions-bangalore.html

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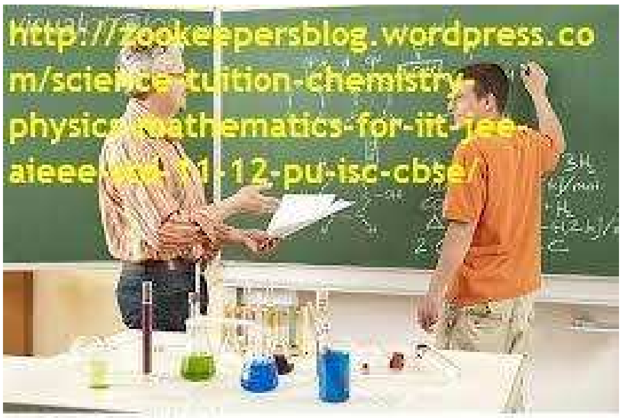


A very polite request :

I wish these e-Books are read only by Boys and Men. Girls and Women, better read something else; learn from somewhere else.

Preface

We all know that in the species "Homo Sapiens ", males are bigger than females. The reasons are explained in standard 10, or 11 (high school) Biology texts. **This shapes or size, influences all of our culture**. Before we recall / understand the reasons once again, let us see some random examples of the influence

Random - 1

If there is a Road rage, then who all fight ? (generally ?). Imagine two cars driven by adult drivers. Each car has a woman of similar age as that of the Man. The cars " touch " or " some issue happens". Who all comes out and fights ? Who all are most probable to drive the cars ?

(Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win)

Random - 2

Heavy metal music artists are all Men. Metallica, Black Sabbath, Motley Crue, Megadeth, Motorhead, AC/DC, Deep Purple, Slayer, Guns & Roses, Led Zeppelin, Aerosmith ….. the list can be in thousands. All these are grown-up Boys, known as Men.

(Men strive for perfection. Men are eager to excel. Men work hard. Men want to win.)

Random - 3

Apart from Marie Curie, only one more woman got Nobel Prize in Physics. (Maria Goeppert Mayer - 1963). So, … almost all are men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 4

The best Tabla Players are all Men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 5

History is all about, which Kings ruled. Kings, their men, and Soldiers went for wars. History is all about wars, fights, and killings by men.

Boys start fighting from school days. Girls do not fight like this

(Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

Random - 6

The highest award in Mathematics, the " Fields Medal " is around since decades. Till date only one woman could get that. (Maryam Mirzakhani - 2014). So, … almost all are men.

(Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women.)

Random - 7

Actor is a gender neutral word. Could the movie like " Top Gun " be made with Female actors ? The best pilots, astronauts, Fighters are all Men.

Random - 8

In my childhood had seen a movie named “ The Tower in Inferno “. In the movie when the tall tower is in fire, women were being saved first, as only one lift was working….

Many decades later another movie is made. A box office hit. “ The Titanic “. In this also …. As the ship is sinking women are being saved. **Men are disposable**. Men may get their turn later…

Movies are not training programs. Movies do not teach people what to do, or not to do. Movies only reflect the prevalent culture. Men are disposable, is the culture in the society. Knowingly, unknowingly, the culture is depicted in Movies, Theaters, Stories, Poems, Rituals, etc. I or you can’t write a story, or make a movie in which after a minor car accident the Male passengers keep seating in the back seat, while the both the women drivers come out of the car and start fighting very bitterly on the road. There has been no story in this world, or no movie made, where after an accident or calamity, Men are being helped for safety first, and women are told to wait.

Random - 9

Artists generally follow the prevalent culture of the Society. In paintings, sculptures, stories, poems, movies, cartoon, Caricatures, knowingly / unknowingly, “ the prevalent Reality “ is depicted. The opposite will not go well with people. If deliberately “ the opposite “ is shown then it may only become a special art, considered as a special mockery.

Random - 10

Men go to “girl / woman’s house” to marry / win, and bring her to his home. That is a sort of winning her. When a boy gets a “ Girl-Friend “, generally he and his friends consider that as an achievement. The boy who “ got / won “ a girl-friend feels proud. His male friends feel, jealous, competitive and envious. Millions of stories have been written on these themes. Lakhs of movies show this. Boys / Men go for “ bike race “, or say “ Car Race “, where the winner “ gets “ the most beautiful girl of the college.




(Men want to excel. Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

Prithviraj Chauhan ‘ went ` to “ pickup “ or “ abduct “ or “ win “ or “ bring “ his love. There was a Hindi movie (hit) song … “ Pasand ho jaye, to ghar se utha laye “. It is not other way round. Girls do not go to Boy’s house or man’s house to marry. Nor the girls go in a gang to “ pick-up “ the boy / man and bring him to their home / place / den.

Random - 11

Rich people; often are very hard working. Successful business men, establish their business (empire), amass lot of wealth, with lot of difficulty. Lots of sacrifice, lots of hard work, gets into this. Rich people's wives had no contribution in this wealth creation. Women are smart, and successful upto the extent to choose the right/rich man to marry. So generally what happens in case of Divorces ? Search the net on " most costly divorces " and you will know. The women;(who had no contribution at all, in setting up the business / empire), often gets in Billions, or several Millions in divorce settlements.

Number 1

Rupert & Anna Murdoch -- $1.7 billion

One of the richest men in the world, **Rupert Murdoch** developed his worldwide media empire when he inherited his father's Australian newspaper in 1952. He married Anna Murdoch in the '60s and they remained together for 32 years, springing off three children.

They split amicably in 1998 but soon Rupert forced Anna off the board of News Corp and the gloves came off. The divorce was finalized in June 1999 when Rupert agreed to let his ex-wife leave with $1.7 billion worth of his assets, $110 million of it in cash. Seventeen days later, Rupert married Wendi Deng, one of his employees.

Ted Danson & Casey Coates -- $30 million

Ted Danson's claim to fame is undoubtedly his decade-long stint as Sam Malone on NBC's celebrated sitcom Cheers . While he did other TV shows and movies, he will always be known as the bartender of that place where everybody knows your name. He met his future first bride Casey, a designer, in 1976 while doing Erhard Seminars Training.

Ten years his senior, she suffered a paralyzing stroke while giving birth to their first child in 1979. In order to nurse her back to health, Danson took a break from acting for six months. But after two children and 15 years of marriage, the infatuation fell to pieces. Danson had started seeing Whoopi Goldberg while filming the comedy, Made in America and this precipitated the 1992 divorce. Casey got $30 million for her trouble.

See https://zookeepersblog.wordpress.com/misandry-and-men-issues-a-short-summary-at-single-place/

See http://skmclasses.kinja.com/save-the-male-1761788732

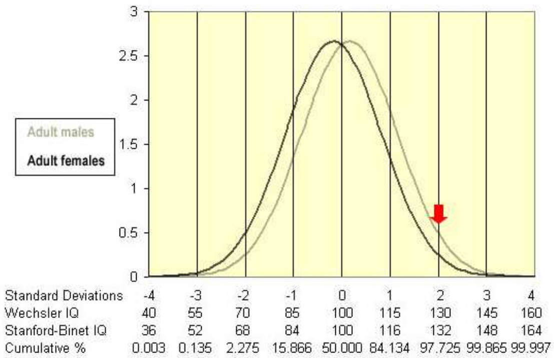
It was Boys and Men, who brought the girls / women home. The Laws are biased, completely favoring women. The men are paying for their own mistakes.

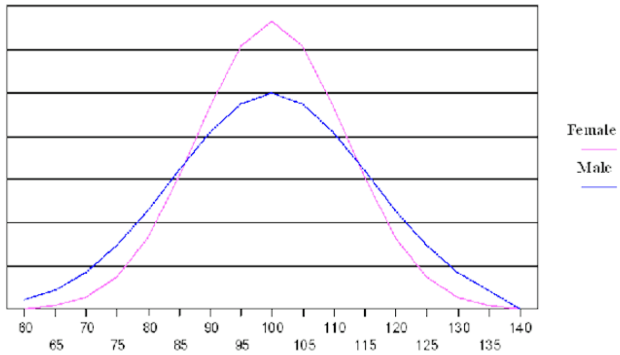
See https://zookeepersblog.wordpress.com/biased-laws/

(Man brings the Woman home. When she leaves, takes away her share of big fortune!)

Random - 12

A standardized test of Intelligence will never be possible. It never happened before, nor ever will happen in future; where the IQ test results will be acceptable by all. In the net there are thousands of charts which show that the intelligence scores of girls / women are lesser. Debates of Trillion words, does not improve performance of Girls.





I am not wasting a single second debating or discussing with anyone, on this. I am simply accepting ALL the results. IQ is only one of the variables which is required for success in life. Thousands of books have been written on " Networking Skills ", EQ (Emotional Quotient), Drive, Dedication, Focus, " Tenacity towards the end goal " … etc. In each criteria, and in all together, women (in general) do far worse than men. Bangalore is known as " ….. capital of India ". [Fill in the blanks]. The blanks are generally filled as " Software Capital ", " IT Capital ", " Startup Capital ", etc. I am member in several startup eco-systems / groups. I have attended hundreds of meetings, regarding " technology startups ", or " idea startups ". These meetings have very few women. Starting up new companies are all " Men's Game " / " Men's business ". Only in Divorce settlements women will take their goodies, due to Biased laws. There is no dedication, towards wealth creation, by women.

Random - 13

Many men, as fathers, very unfortunately treat their daughters as " Princess ". Every " non-performing " woman / wife was " princess daughter " of some loving father. Pampering the girls, in name of " equal opportunity ", or " women empowerment ", have led to nothing.

See http://skmclasses.kinja.com/progressively-daughters-become-monsters-1764484338

See http://skmclasses.kinja.com/vivacious-vixens-1764483974

There can be thousands of more such random examples, where " Bigger Shape / size " of males have influenced our culture, our Society. **Let us recall the reasons**, that we already learned in standard 10 - 11, Biology text Books. In humans, women have a long gestation period, and also spends many years (almost a decade) to grow, nourish, and stabilize the child. (Million years of habit) Due to survival instinct Males want to inseminate. Boys and Men fight for the " facility (of womb + care) " the girl / woman may provide. Bigger size for males, has a winning advantage. Whoever wins, gets the " woman / facility ". The male who is of " Bigger Size ", has an advantage to win.... Leading to Natural selection over millions of years. In general " Bigger Males "; the " fighting instinct " in men; have led to wars, and solving tough problems (Mathematics, Physics, Technology, startups of new businesses, Wealth creation, Unreasonable attempts to make things [such as planes], Hard work ....)

So let us see the IIT-JEE results of girls. Statistics of several years show that there are around 17, (or less than 20) girls in top 1000 ranks, at all India level. Some people will yet not understand the performance, till it is said that ... year after year we have around 980 boys in top 1000 ranks. Generally we see only 4 to 5 girls in top 500. In last 50 years not once any girl topped in IIT-JEE advanced. Forget about Single digit ranks, double digit ranks by girls have been extremely rare. It is all about " good boys ", " hard working ", " focused ", "**Bel-esprit**" **boys**.

In 2015, Only 2.6% of total candidates who qualified are girls (upto around 12,000 rank). while 20% of the Boys, amongst all candidates qualified. The Total number of students who appeared for the exam were around 1.4 million for IIT-JEE main. Subsequently 1.2 lakh (around 120 thousands) appeared for IIT-JEE advanced.

IIT-JEE results and analysis, of many years is given at https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/

In Bangalore it is rare to see a girl with rank better than 1000 in IIT-JEE advanced. We hardly see 6-7 boys with rank better than 1000. Hardly 2-3 boys get a rank better than 500.

See http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html

Thousands of people are exposing the heinous crimes that Motherly Women are doing, or Female Teachers are committing. See https://www.facebook.com/WomenCriminals/

Some Random Examples must be known by all

It is extremely unfortunate that the " woman empowerment " has created. This is the kind of society and women we have now. I and many other sensible Men hate such women. Be away from such women, be aware of reality.

Mother Admits On Facebook to Sleeping with 15 Yr Old Son, They Have a Baby Together - Alwayzturntup

Sometimes it hard to believe w From Alwayzturntup

ALWAYZTURNTUP.ME

'Sex with my son is incredible - we're in love and we want a baby'

Ben Ford, who ditched his wife when he met his mother Kim West after 30 years, claims what the couple are doing 'isn't incest'

MIRROR.CO.UK

Former Shelbyville ISD teacher who had sex with underage student gets 3 years in prison

After a two day break over the weekend, A Shelby County jury was back in the courtroom looking to conclude the trial of a former Shelbyville ISD teacher who had...

KLTV.COM | BY CALEB BEAMES

Woman sent to jail for the rest of her life after raping her four grandchildren is described as the 'most evil person' the judge has ever seen

Edwina Louis rape...

See More

Woman sent to jail for raping her four grandchildren

A Ohio grandmother has been sentenced to four consecutive life terms after being found guilty of the rape of her own grandchildren. Edwina Louis, 53, will spend the rest of her life behind bars.

DAILYMAIL.CO.UK

http://www.thenativecanadian.com/.../eastern-ontario-teacher-...

The N.C. Chronicles.: Eastern Ontario teacher charged with 36 sexual offences

anti feminism, Child abuse, children's rights, Feminist hypocrisy,

THENATIVECANADIAN.COM | BY BLACKWOLF

Hyd woman kills newborn boy as she wanted daughter - Times of India

Having failed to bear a daughter for the third time, a shopkeeper's wife slit the throat of her 24day-old son with a shaving blade and left him to die in a street on Tuesday night.Purnima's first child was a stillborn boy, followed by another boy born five years ago.

TIMESOFINDIA.INDIATIMES.COM

Montgomery's son, Alan Vonn Webb, took the stand and was a key witness in her conviction.

"I want to see her placed somewhere she can never do that to children ...

See More

Woman sentenced to 40 years in prison for raping her children

A Murfreesboro mother found guilty of raping her own children learned her fate on Wednesday.

WAFF.COM | BY DENNIS FERRIER

gentler sex? Violence against men.'s photo.

Women, the gentler sex? Violence against men. Like Page

April 8 at 1:38am ·

In fact, the past decade has seen a dramatic increase in the number of incidents of women raping and sexually assaulting boys and men. On May 2014, Jezebel repo...

End violence against women . . . .

North Carolina Grandma Eats Her Daughter's New Born Baby After Smoking Bath Salts

Henderson, North Carolina– A North Carolina grandmother of 4 and recovering drug addict, is now in custody after she allegedly ate her daughter's newborn baby....

AZ-365.TOP

28-Year-Old Texas Teacher Accused of Sending Nude Picture to 14-Year-Old Former Student

BREITBART.COM

http://latest.com/.../attractive-girl-gang-lured-men-alleywa.../

Attractive Girl Gang Lured Men Into Alleyways Where Female Body Builder Would Attack Them

A Mexican street gang made up entirely of women has been accused of using their feminine wiles to lure men into alleyways and then beating them up and...

LATEST.COM

http://www.wfmj.com/.../youngstown-woman-convicted-of-raping-...

Youngstown woman convicted of raping a 1 year old is back in jail

A Youngstown woman who went to prison for raping a 1-year-old boy fifteen years ago is in trouble with the law again.

WFMJ.COM

End violence against women . . . .

Women are raping boys and young men

Rape advocacy has been maligned and twisted into a political agenda controlled by radicalized activists. Tim Patten takes a razor keen and well supported look into the manufactured rape culture and...

AVOICEFORMEN.COM | BY TIM PATTEN

Bronx Woman Convicted of Poisoning and Drowning Her Children

Lisette Bamenga researched methods on the Internet before she killed her son and daughter in 2012.

NYTIMES.COM | BY MARC SANTORA

A Russian-born newlywed slowly butchered her German husband — feeding strips of his flesh to their dog until he took his last breath. Svetlana Batukova, 46, was...

See More

She killed her husband and then fed him to her dog: police

A Russian-born newlywed butchered her German hubby — and fed strips of his flesh to her pooch, authorities said. Svetlana Batukova offed Horst Hans Henkels at their...

NYPOST.COM

Daily Mail
January 15, 2015 ·

Mother charged with rape and sodomy of her son's 12-year-old friend

Mom, 30, 'raped and had oral sex with her son's 12-year-old friend'

Nicole Marie Smith, 30, (pictured) of St Charles County, Missouri, has been jailed after she allegedly targeted the 12-year-old boy at her home.

DAILYM.AI

April 4 at 4:48am ·

Female prison officers commit 90pc of sex assaults on male teens in US juvenile detention centres

Lawsuit in Idaho highlights the prevalence of sexual victimization of juvenile offenders.

IBTIMES.CO.UK | BY NICOLE ROJAS

This mother filmed herself raping her own son and then sold it to a man for $300. The courts just decide her fate. When you see what she got, you're going to be outraged.

Mother Who Filmed Herself Raping Her 1-Year-Old Son Receives Shocking Sentence

"...then used the money to buy herself a laptop..."

AMERICANEWS.COM

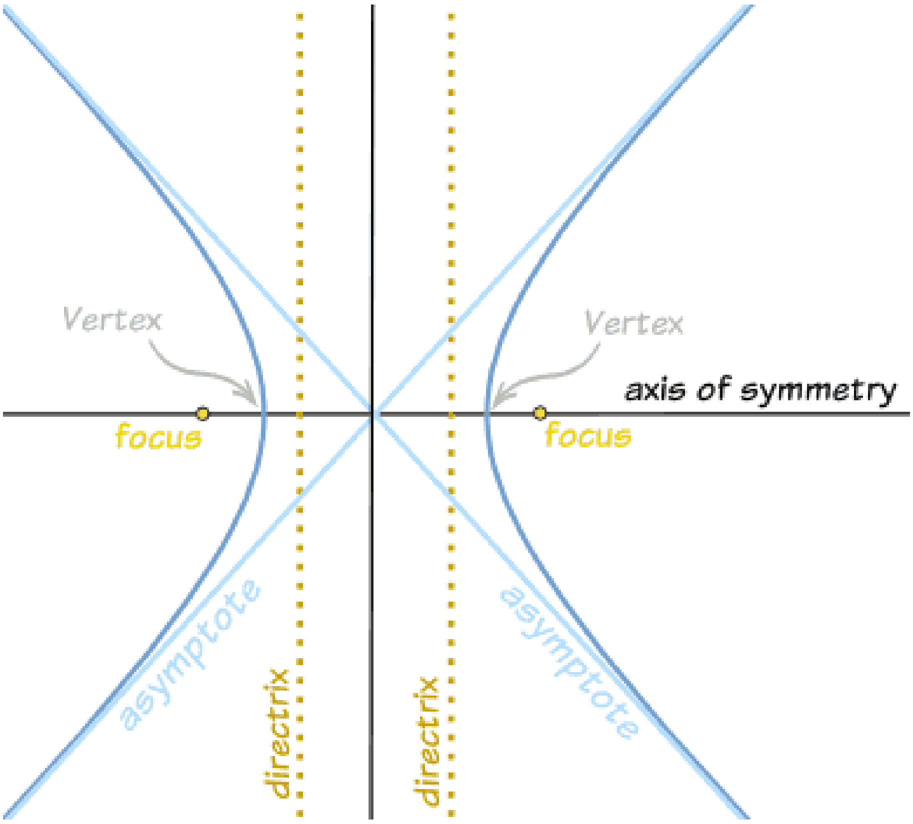
In several countries or rather in several regions of the world, family system has collapsed, due to bad nature and naughty acts of women. Particularly in Britain, and America, almost 50% people are alone, lonely, separated, divorced or failed marriages. In 2013, 48% children were born out of wedlock. It was projected that by 2016, more than 51% children will be born, to unmarried mothers. In these developed countries " paternity fraud " by women, are close to 20%. You can see several articles in the net, and in wikipedia etc. This means 1 out of 5 children are calling a wrong man as dad. The lonely, alone “ mothers " are frustrated. They see the children as burden. Love in the Society in general is lost, long time ago. The types of “ Mothers “ and “ Women “ we have now …………

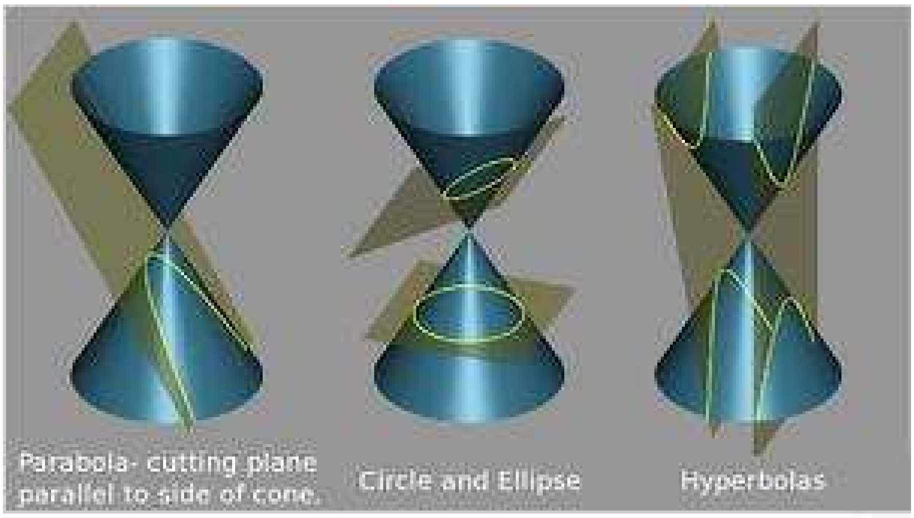
By now if you have assumed that Indian women are not doing any crime then please become friends with MRA Guri https://www.facebook.com/profile.php?id=100004138754180

He has dedicated his life to expose Indian Criminals

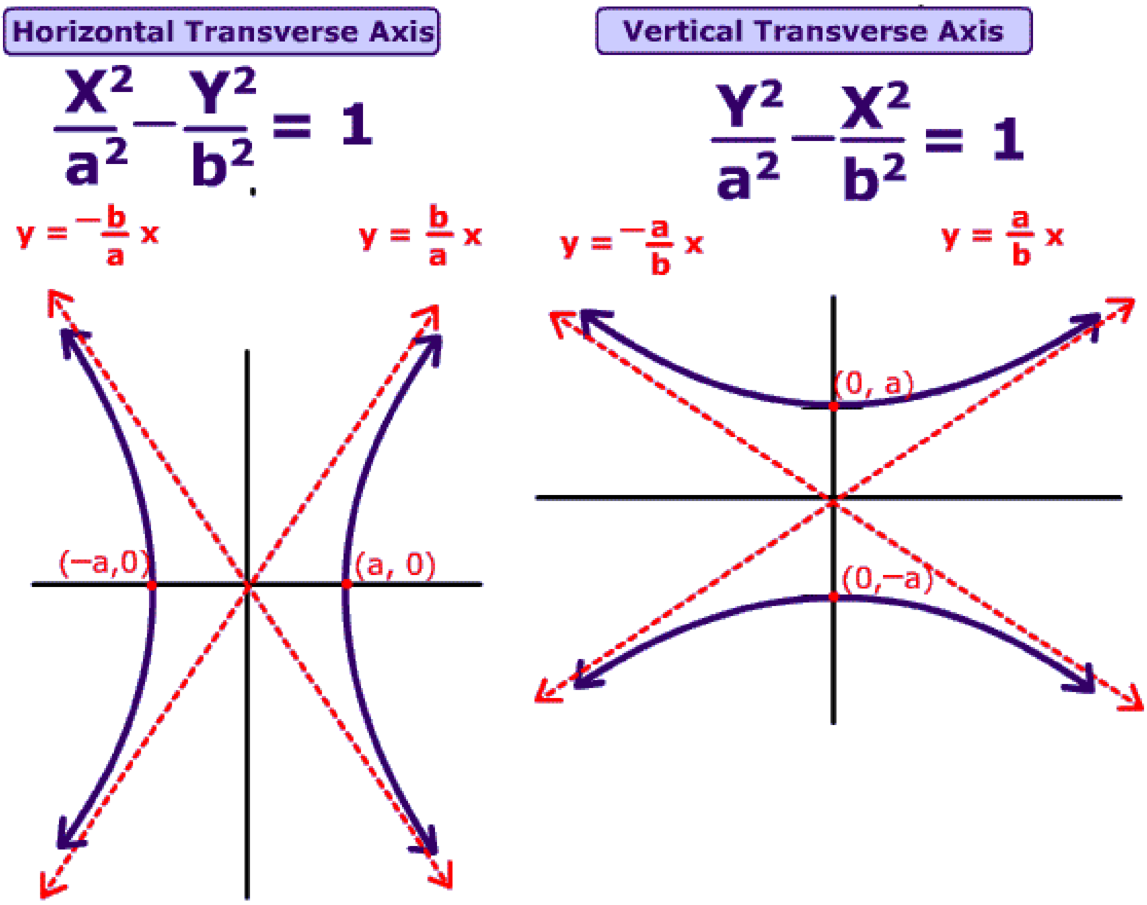
Professor Subhashish Chattopadhyay

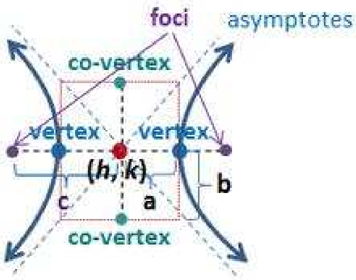
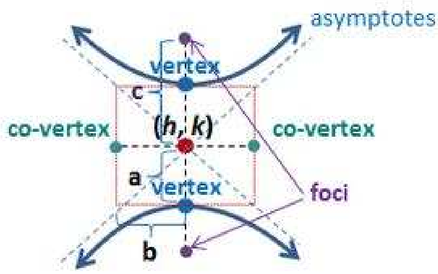
Spoon Feeding Series - Hyperbola

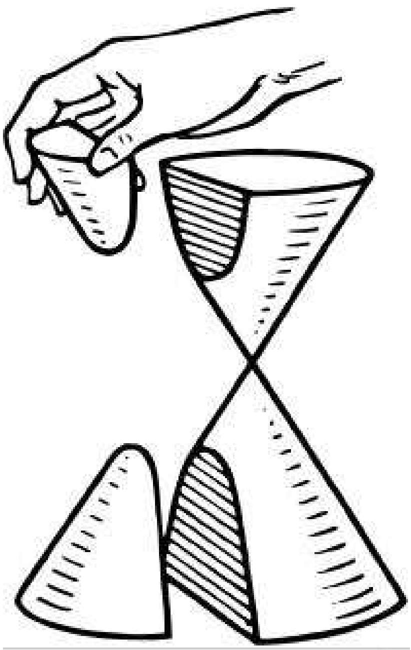




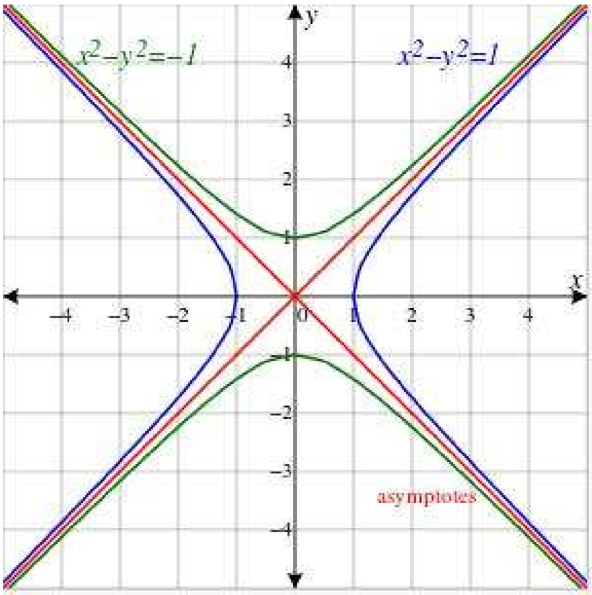
Three major types of conic sections.


Horizontal Transverse Axis
$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$
$y = -\frac{b}{a}x$
$y = \frac{b}{a}x$
(−a,0)
(a, 0)
Vertical Transverse Axis
$\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$
$y = -\frac{a}{b}x$
$y = \frac{a}{b}x$
(0, a)
(0,−a)

Horizontal Hyperbola (**x^2 comes first**)	**Vertical Hyperbola** (**y^2 comes first**)
At (0, 0): $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	At (0, 0): $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
General: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	**General:** $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
$a^2+b^2=c^2$	$a^2+b^2=c^2$
Center: (h,k) **Foci:** $(h\pm c,k)$	**Center:** (h,k) **Foci:** $(h,k\pm c)$
Vertices: $(h\pm a,k)$ **Co-Vertices:** $(h,k\pm b)$	**Vertices:** $(h,k\pm a)$ **Co-Vertices:** $(h\pm b,k)$
Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$	**Asymptotes:** $y-k=\pm\frac{a}{b}(x-h)$
	



A hyperbola is an open curve with two branches, the intersection of a plane with both halves of a double cone. The plane does not have to be parallel to the axis of the cone; the hyperbola will be symmetrical in any case.



Here $a = b = 1$ giving the unit hyperbola in blue and its conjugate hyperbola in green, sharing the same red asymptotes.

Example: Write the equation in standard form of $4x^2 - 16y^2 = 64$. ***Find the foci and vertices of the hyperbola.***

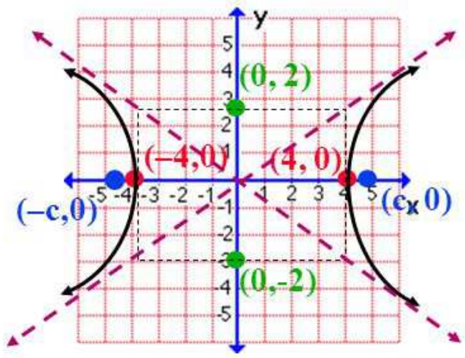
Divide both side of the equation by 64 to get the standard form:

$$\frac{4x^2}{64} - \frac{16y^2}{64} = \frac{64}{64}$$

Simplify...

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

That means $a = 4 \quad b = 2$



Use $c^2 = a^2 + b^2$ ***to find*** c.

$$c^2 = 4^2 + 2^2$$

$$c^2 = 16 + 4 = 20$$

$$c = \sqrt{20} = 2\sqrt{5}$$

Vertices: $(-4, 0)$ *and* $(4, 0)$

Foci: $(-2\sqrt{5}, 0)$ *and* $(2\sqrt{5}, 0)$

A **hyperbola** sort of looks like two parabolas that point at each other, and is the set of points whose distances from two fixed points (the foci) inside the ellipse is always the same, $d_1 - d_2 = 2a$. The distance **2*a*** is called the **focal radii distance**, **focal constant,** or **constant difference**, and ***a*** is the distance between the center of the hyperbola to a **vertex**.

The equation of a "horizontal" hyperbola (as shown below) that is centered on the origin **(0, 0)** is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The length of the axis in which the hyperbola lies (called the **transverse axis**) is **2*a***, and this is along the ***x*-axis** for a horizontal hyperbola. Again, the distance from the center of the hyperbola to a vertex is ***a***, so the vertices are at $(\pm a, 0)$.

The length of the **conjugate axis** is **2*b***, and note that ***a*** does **not have to be bigger** than ***b***, like it does for an ellipse. (The distance from the center of the hyperbola to a co-vertex is ***b***). Also note where the ***b*** is not on the hyperbola; it is on what we call the **central rectangle** (or **fundamental rectangle**) of the hyperbola (whose **diagonals are asymptotes** for the hyperbola). So the **conjugate axis** is along the ***y*-axis** for a horizontal hyperbola, and the co-vertices are at $(0, \pm b)$.

The **asymptotes** for a horizontal hyperbola centered at (0, 0) are $y = \pm\frac{b}{a}x$ ($\pm\frac{b}{a}$ are the **slopes**, or the square root of what's under the ***y*** over the square root of what's under the ***x***).

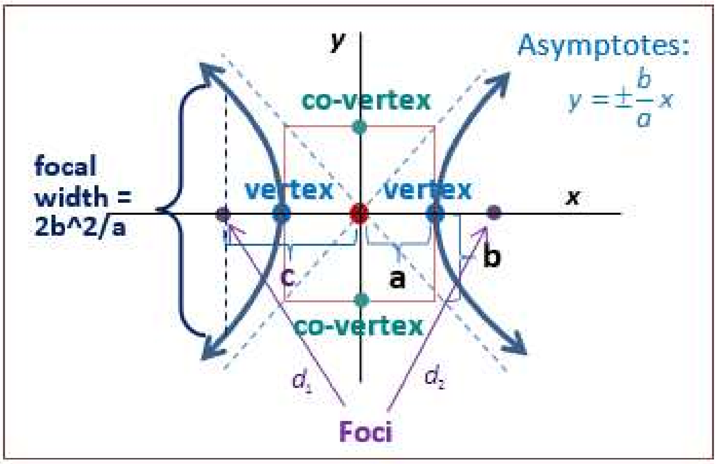
The asymptotes are the **diagonals** of the central rectangle of the hyperbola.

The focuses or **foci** always lie inside the curves on the **major axis**, and the distance from the center to a focus is ***c***. So the foci are at $(\pm c, 0)$ for a horizontal hyperbola (like an ellipse!), and it turns out that $a^2 + b^2 = c^2$. (I like to remember that you always use the different sign for this equation: since ellipses have a **plus sign** in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, they have a **minus sign** in $a^2 - b^2 = c^2$; since hyperbolas have a **minus sign** in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, they have a **plus sign** in $a^2 + b^2 = c^2$.)

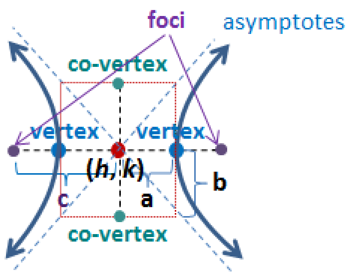
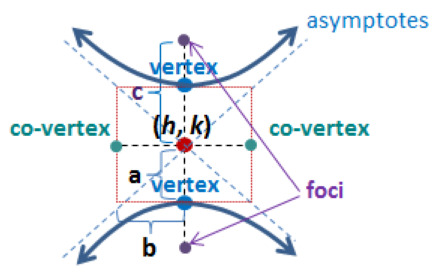
Sometimes you will be asked to get the **eccentricity** of an hyperbola $\frac{c}{a}$, which is a measure of how "straight" or "stretched" the hyperbola is.

Note also that, like for an ellipse, the **focal width** (**focal chord**, or **focal rectum**) of an ellipse is $\frac{2b^2}{a}$; this the distance perpendicular to the major axis that goes through the focus.

Here is a **horizontal hyperbola**; we will also look at vertical and transformed hyperbolas below.



Here are the two different “directions” of hyperbolas and the generalized equations for each:

Horizontal Hyperbola (**x^2 comes first**)	**Vertical Hyperbola** (**y^2 comes first**)
At (0, 0): $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **General:** $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $a^2 + b^2 = c^2$ **Center:** (h, k) **Foci:** $(h \pm c, k)$ **Vertices:** $(h \pm a, k)$ **Co-Vertices:** $(h, k \pm b)$ **Asymptotes:** $y - k = \pm \frac{b}{a}(x-h)$ 	At (0, 0): $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ **General:** $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ $a^2 + b^2 = c^2$ **Center:** (h, k) **Foci:** $(h, k \pm c)$ **Vertices:** $(h, k \pm a)$ **Co-Vertices:** $(h \pm b, k)$ **Asymptotes:** $y - k = \pm \frac{a}{b}(x-h)$ 

Remember, for the conic to be a hyperbola, the coefficients of the x^2 and y^2 **must have different signs.**

The equation of the tangent at (x_1 , y_1) can be obtained by replacing x^2 by xx_1 , y^2 by yy_1 , x by (x + x_1)/2 , y by (y + y_1)/2 and xy by (xy_1 + x_1y)/2

Parametric form The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

Slope form The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

Director circle is the locus of the point from which tangents drawn to the hyperbola are perpendicular. Or in other words, Locus of the point where Perpendicular tangents meet.

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the equation of the director circle is $x^2 + y^2 = a^2 - b^2$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$SS_1 = T^2$$

where $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$

Equation of Normal

Point form The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2.$$

Parametric form The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2.$$

Slope form The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}.$$

The coordinates of the points of contact are

$$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2 m^2}}, \mp \frac{mb^2}{\sqrt{a^2 - b^2 m^2}} \right)$$

CHORD WITH A GIVEN MID POINT

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $P(x_1, y_1)$ as its middle point is given by $T = S_1$ where

$$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 \text{ and } S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1.$$

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$, where

$$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$$

Question

The equation of the hyperbola, referred to its axes as axes of coordinates, given that the distances of one of its vertices from the foci are 9 and 1 units, is

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (b) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(c) $\frac{x^2}{16} - \frac{y^2}{9} = -1$ (d) none of these

Solution

(a). Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(1)$$

Its vertices are $A(a, 0)$ and $A'(-a, 0)$ and foci are $S(ae, 0)$ and $S'(-ae, 0)$.

Given : $S'A = 9$ and $SA = 1$

$\Rightarrow \quad a + ae = 9$ and $ae - a = 1$

$\Rightarrow \quad a(1 + e) = 9$ and $a(e - 1) = 1$

$$\Rightarrow \frac{a(1+e)}{a(e-1)} = \frac{9}{1} \Rightarrow 1 + e = 9e - 9 \Rightarrow e = \frac{5}{4}.$$

$$\because \; a(1 + e) = 9, \; \therefore \; a\left(1 + \frac{5}{4}\right) = 9 \Rightarrow a = 4.$$

$$\text{Also, } b^2 = a^2(e^2 - 1) = 16\left(\frac{25}{16} - 1\right) = 9.$$

Thus, from (1), equation of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

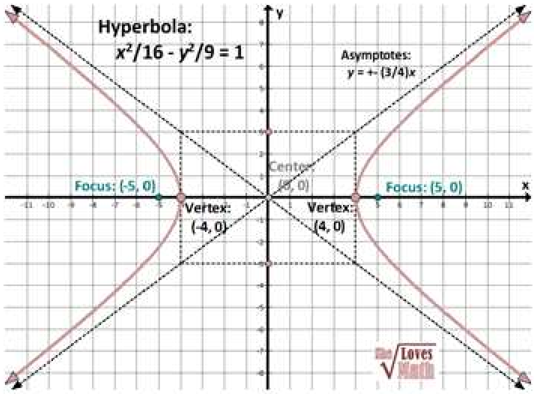
Question

Identify the **center**, **vertices**, **foci**, and **equations of the asymptotes** for the following **hyperbolas**; then **graph**: (a) $9x^2 - 16y^2 - 144 = 0$ (b) $\frac{(y+3)^2}{4} - \frac{(x-2)^2}{36} = 1$.

It's typically easier to **graph the hyperbola first**, and then answer the questions.

Graph	Math/Notes

(a) $9x^2-16y^2-144=0$, which is $\frac{x^2}{16}-\frac{y^2}{9}=1$



Domain: $(-\infty,4]\cup[4,\infty)$ **Range:** $(-\infty,\infty)$

Notice that the vertices and foci lie are along the **horizontal line** y = 0, and the length of the **transverse axis** is **2*a***, or 8. The length of the **conjugate axis** is **2*b***, or 6. √

We first need to get our equation into the form of hyperbola by adding 144 to both sides, and then dividing all terms by 144: $\frac{9x^2}{144}-\frac{16y^2}{144}=\frac{144}{144}$, or $\frac{x^2}{16}-\frac{y^2}{9}=1$. We will use the equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, since the ***x*** comes first (**horizontal**).

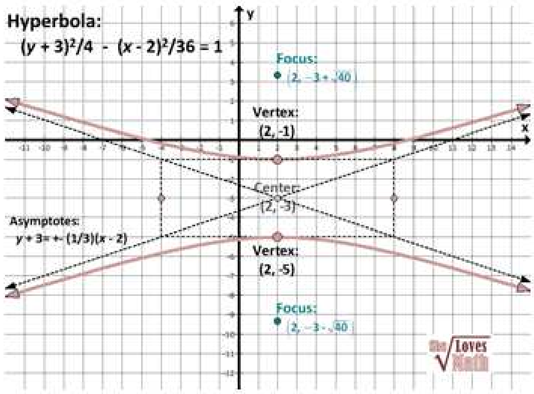
This would make $a^2=16$, so $a=4$. Since the hyperbola's **center** is **(0, 0)**, the **vertices** are **(–4, 0)** and **(4, 0)**.

$b^2=9$, so $b=3$, so the co-vertices are **(0, –3)** and **(0, 3)**. Now we can construct our central rectangle; we use *a* and *b* to create it.

Now let's find the foci: $c^2=a^2+b^2=9+16=25$. So *c* = 5, , and the **foci** are $(\mathbf{0},\pm\mathbf{5})$.

The equation of the **asymptotes** (which go through the corners of the central rectangle) are $y-k=\pm\frac{b\text{ (rise)}}{a\text{ (run)}}(x-h)$, or $y=\pm\frac{3}{4}x$. (Remember to use the square root of what's under the ***y*** for the numerator of the slope, and the square of what's under the ***x*** for the denominator.)

(b) $\frac{(y+3)^2}{4}-\frac{(x-2)^2}{36}=1$



Domain: $(-\infty,\infty)$ **Range:** $(-\infty,-5]\cup[5,\infty)$

Notice that the vertices and foci are along the **vertical line** x = 2, and the length of the **transverse axis** is **2*a***, or 4. The length of the **conjugate axis** is **2*b***, or 12. √

We will use equation: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, since the ***y*** comes first (**vertical**).

We see that the **center** of the hyperbola is at (2, –3), so we can first plot that point.

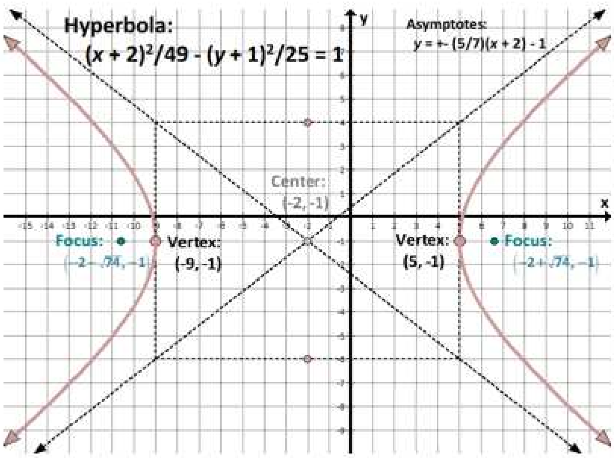
$a^2=4$, so $a=2$. Since the center is (2, –3), the **vertices** are (2, –3 – 2) and (2, –3 + 2), or **(2 , –5)** and **(2, –1)**.
$b^2=36$, so $b=6$, so the **co-vertices** are (2 – 6 , –3) and (2 + 6 , –3) or **(–4 , –3)** and **(8, –3)**.

Now let's find the foci: $c^2=a^2+b^2=4+36=40$. So $c=\sqrt{40}$ (or $2\sqrt{10}$), and the **foci** are $\left(\mathbf{2},-\mathbf{3}\pm\sqrt{\mathbf{40}}\right)$.

The equation of the **asymptotes** (which go through the corners of the **central rectangle**) are $y-k=\pm\frac{a\text{ (rise)}}{b\text{ (run)}}(x-h)$, or $y+3=\pm\frac{2}{6}(x-2)$ **or** $y+3=\pm\frac{1}{3}(x-2)$. (Remember to use the square root of what's under the ***y*** for the numerator of the slope, and the square of what's under the ***x*** for the denominator.)

Question

Identify the the **center**, **vertices**, **foci**, and **equations of the asymptotes** for the following **hyperbola**; then **graph**: $49y^2 - 25x^2 + 98y - 100x + 1174 = 0$.

Graph	Math/Notes
$49y^2 - 25x^2 + 98y - 100x + 1174 = 0$ $49y^2 + 98y - 25x^2 - 100x = -1174$ $49(y^2 + 2y) - 25(x^2 + 4x) = -1174$ $49(y^2 + 2y + \_\_) - 25(x^2 + 4x + \_\_) = -1174 + \_\_$ $49(y^2 + 2y + (1)^2) - 25(x^2 + 4x + (2)^2) = -1174 + 49(1)^2 - 25(2)^2$ $49(y+1)^2 - 25(x+2)^2 = -1225$ $\frac{(x+2)^2}{49} - \frac{(y+1)^2}{25} = 1$  **Domain:** $(-\infty, -9] \cup [5, \infty)$ **Range:** $(-\infty, \infty)$	We need to first **complete the square** so we can get the equation in hyperbola form. Put all the ***x***'s and ***y***'s together with the constant term on the other side. Watch the negative signs; it turns to be a horizontal hyperbola, with the coefficient of the x^2 being positive (we ended up dividing all terms by -1225). We will use equation: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, since ***x*** comes first (**horizontal hyperbola**). We see that the **center** of the hyperbola is at $(-2, -1)$, so we can first plot that point. $a^2 = 49$, so $a = 7$. Since the center is $(-2, -1)$, the **vertices** are $(-2-7, -1)$ and $(-2+7, -1)$, or $\mathbf{(-9, -1)}$ and $\mathbf{(5, -1)}$. $b^2 = 25$, so $b = 5$, so the **co-vertices** are $(-2, -1-5)$ and $(-2, -1+5)$ or $\mathbf{(-2, -6)}$ and $\mathbf{(-2, 4)}$. Now let's find the foci: $c^2 = a^2 + b^2 = 49 + 25 = 74$. $c = \sqrt{74}$, and the **foci** are $\left(\mathbf{-2 \pm \sqrt{74}, -1}\right)$. The equation of the **asymptotes** (which go through the corners of the **central rectangle**) are $y - k = \pm \frac{b \text{ (rise)}}{a \text{ (run)}}(x - h)$, or $y + 1 = \pm \frac{5}{7}(x+2)$. √

Find the equation of the **hyperbola** where the **difference of the focal radii** is 6, and the **endpoints of the conjugate axis** are (-2, 8) and (-2, -2).

Solution:

We probably don't even need to graph this hyperbola, since we're basically given what ***a*** and ***b*** are. Remember that the **difference of the focal radii** is **2*a***, so ***a*** = 3.

Since the endpoints of the conjugate axis are along a vertical line, we know that the hyperbola is **horizontal**, and the **co-vertices** are (-2, 8) and (-2, -2). From this information, we can get the center (midpoint between the co-vertices), which is (-2, 3) and the length of the minor axis (2*b*), which is 10. So *b* = 5. (Draw the points first if it's difficult to see).

So the equation of the ellipse is $\frac{(x+2)^2}{9} - \frac{(y-3)^2}{25} = 1$.

Question

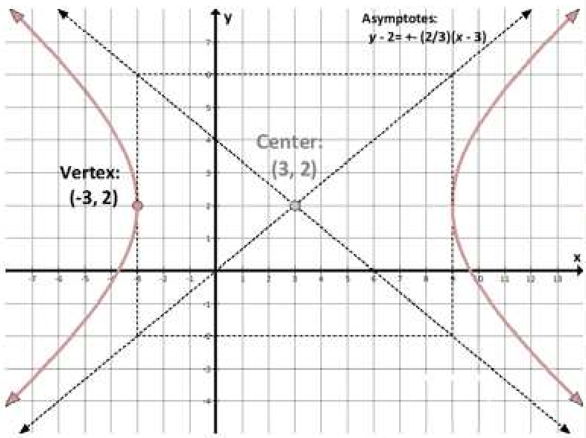
Find the equation of the **hyperbola** where one of the **vertices** is at $(-3, 2)$, and the **asymptotes** are $y - 2 = \pm\frac{2}{3}(x - 3)$.

Solution:

Let's try to graph this one, since it's hard to tell what we know about it!

We can see from the equation of the asymptotes that the **center** of the hyperbola is $(3, 2)$.

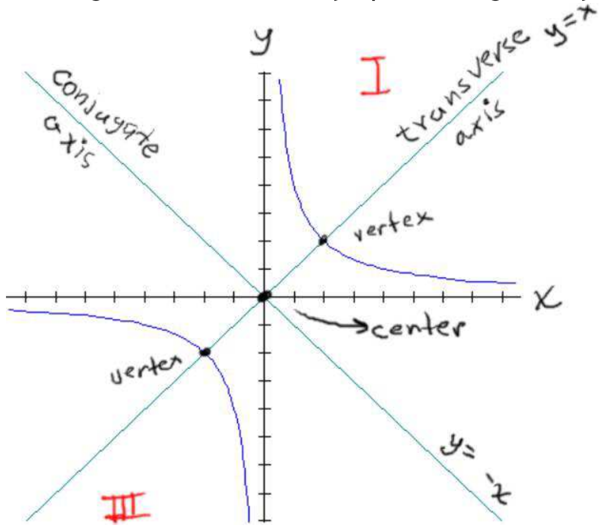
Then we'll graph this **center** and also graph the **vertex** that is given to see that the hyperbola is **horizontal**:

Graph	Math/Notes
 	Now that we know the hyperbola is **horizontal** and we have the **center** and **one vertex**, we can see that ***a*** (difference between center and vertex) is **6.** So far then we have: $\frac{(x-3)^2}{6^2} - \frac{(y-2)^2}{b^2} = 1$ We also see from the asymptotes equation that their **slope** is $\pm\frac{2}{3}$. We actually don't even need to draw them, since we know in our case, since it's a horizontal hyperbola, we'll have $y - 2 = \pm\frac{b\text{ (rise)}}{6\text{ (run)}}(x-3)$ (**rise** is the square root of what's under the ***y***; **run** is the square root of what's under the ***x***) from the equation of the hyperbola above. So now we can set up a **proportion** for the asymptote slopes: $\frac{b}{6} = \frac{2}{3}$; by cross multiplying, we get ***b* = 4.** So the equation of the hyperbola is: $\frac{(x-3)^2}{6^2} - \frac{(y-2)^2}{4^2} = 1$, or $\frac{(x-3)^2}{36} - \frac{(y-2)^2}{16} = 1$. √

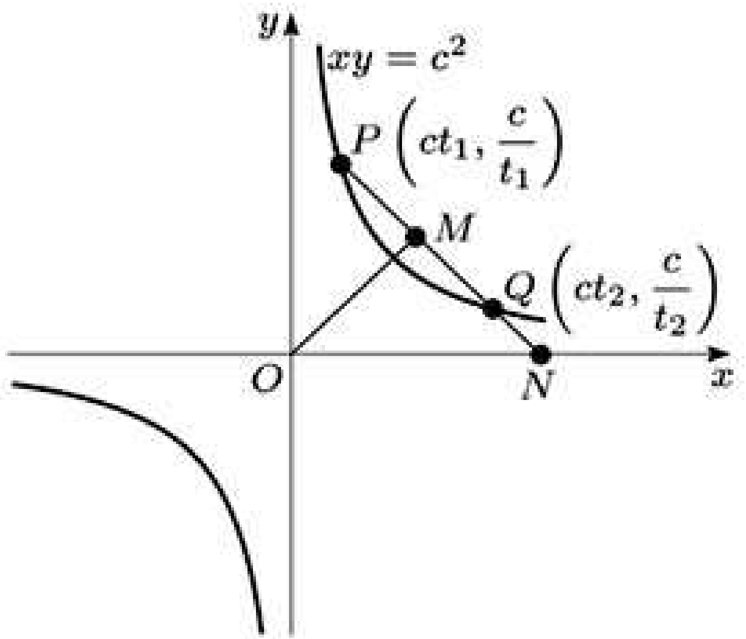
Rectangular Hyperbola

Eccentricity is √2   A hyperbola is said to be a rectangular hyperbola if its asymptotes are at right angles.

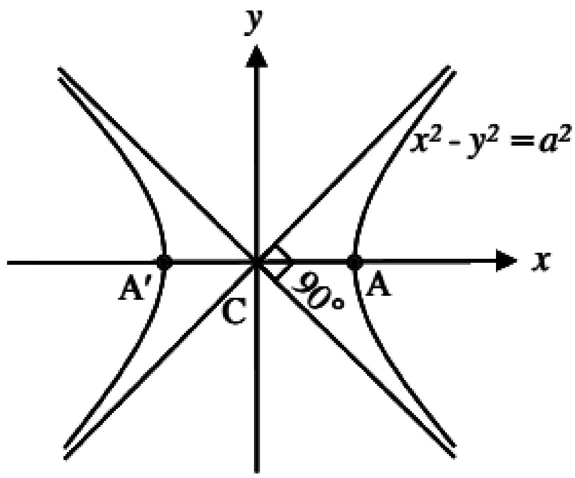
The angle between the asymptotes is given by 2 $\tan^{-1}$ (b/a)



y = 1/x



Another form of Rectangular Hyperbola

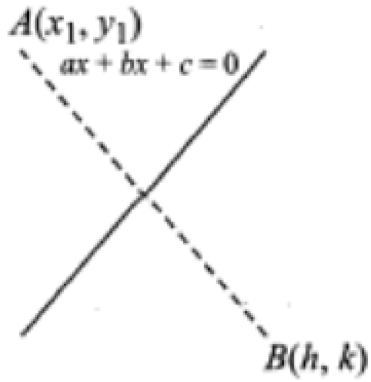


Before we discuss examples and problems let us see the all the formulae

Image of a point

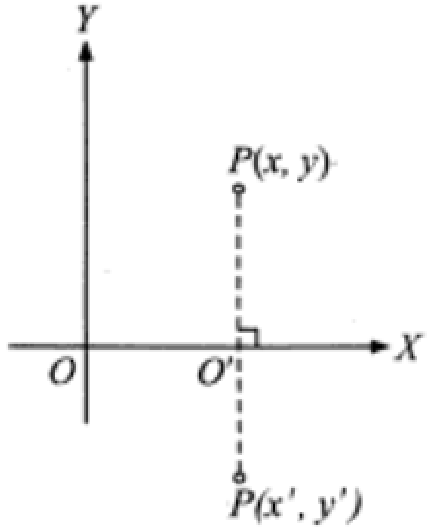
The image of a point with respect to the line mirror. The image of $A(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(h, k)$ given by,

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$



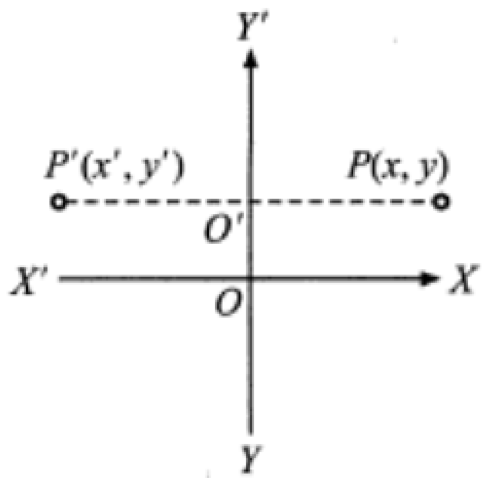
The image of a point with respect to x-axis: Let $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the x-axis, then

$x' = x$ and $y' = -y$, ($\because$ O' is the mid point of PP')



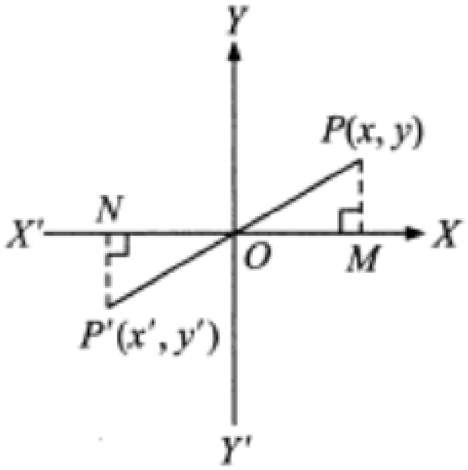
The image of a point with respect to y-axis: $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the y-axis, then

$x' = -x$ and $y' = y$ ($\because$ O' is the mid point of PP')



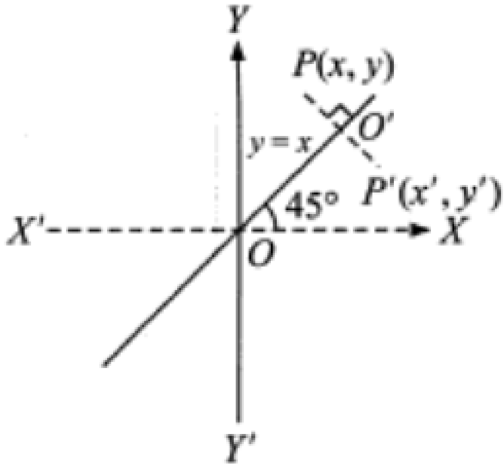
The image of a point with respect to the origin: Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection through the origin, then

$x' = -x$ and $y' = -y$ ($\because$ O is the mid-point of PP')

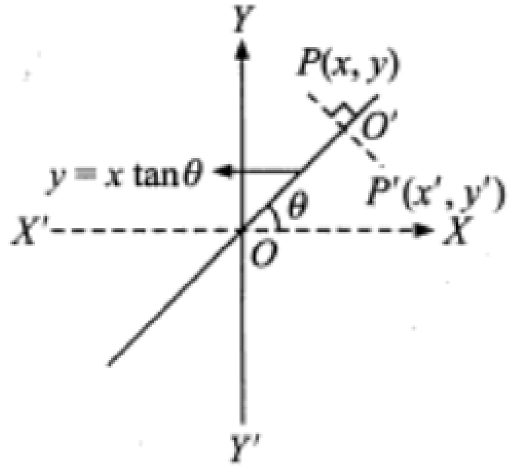


The image of a point with respect to the line $y = x$: Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection in the line $y = x$, then,

$x' = y$ and $y' = x$ ($\because$ O' is the mid-point of PP')

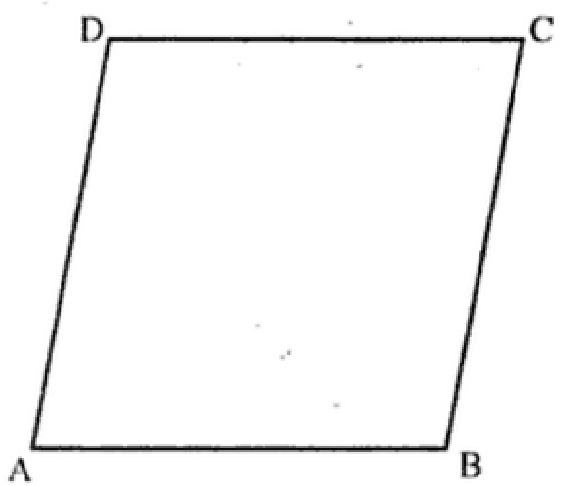


The image of a point with respect to the line $y = x \tan\theta$: Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection in the line $y = x \tan\theta$, then,



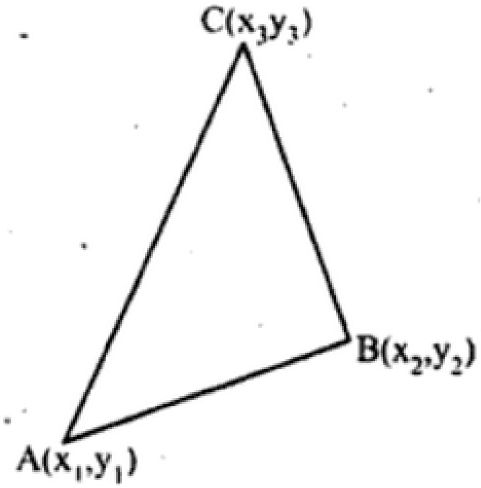
$x' = x \cos 2\theta + y \sin 2\theta$
$y' = x \sin 2\theta - y \cos 2\theta$,
($\because$ O' is the mid-point of PP')

A Rhombus is made by distorting a square



All four sides are equal. So AB = BC = CD = DA

Area of a Triangle



The area of a triangle, the coordinates of whose vertices are (x_1, y_1) (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

or

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition of colinearity of 3 points

Three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear if

i) Area of triangle ABC = 0 i.e.,

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

or

ii) AB+BC=AC (or) AC+BC=AB (or) AC+AB=BC

In some cases a problem can be solved just by observation. Meaning the above determinant need not be evaluated.

The points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ are

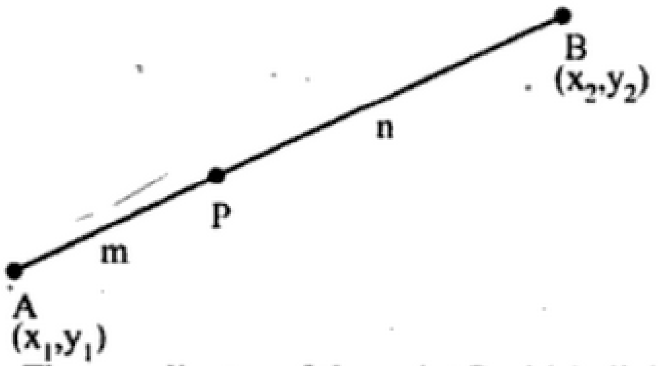
(a) vertices of an equilateral triangle
(b) concyclic
(c) vertices of a right angled triangle
(d) none of these

Ans. (d)

Solution As the given points lie on the line $x + y = a + b + c$, they are collinear.

-

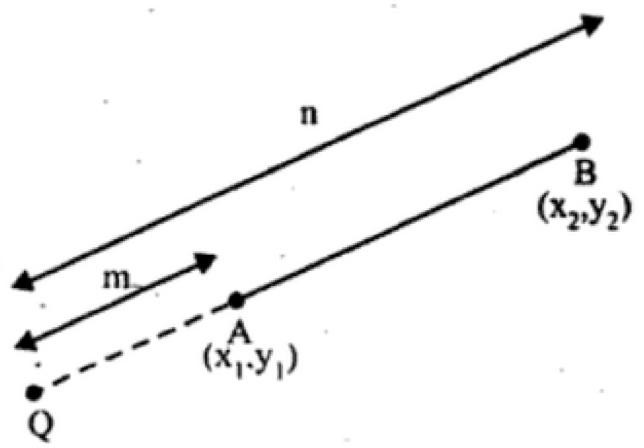
Section formula Internal Division



The coordinates of the point P which divides the line segment joining the points $A(x_1\ y_1)$ and $B(x_2\ y_2)$ internally in the ratio m:n are given by

$$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

Section formula External Division can have Two formulae. Depending on from which external side the division is being done



Here the external point Q is on the side of A

If m is the distance from A then m gets multiplied to coordinates of opposite point i.e.

B(x_2 , y_2)

The coordinates of the point Q which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio m:n are given by

$$Q = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right)$$

Note:

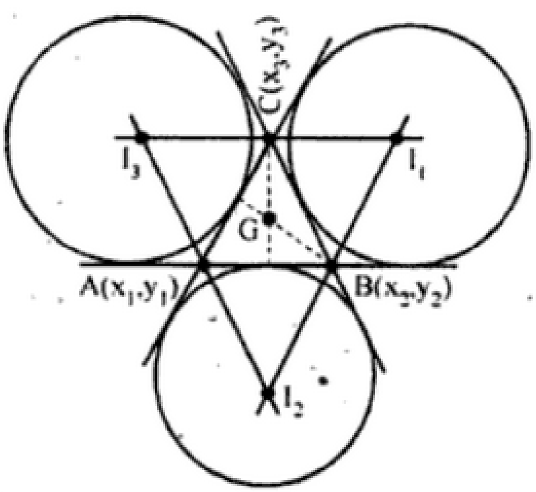
i) If P is the mid point of AB, then the coordinate of P is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

ii) The co-ordinate of any point on AB can be written as $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right)$

Coordinates of the centroid, in-centre and ex-centres of a triangle

Let $A(x_1,y_1)$ $B(x_2\ y_2)$ and $C(x_3\ y_3)$ be the three vertices of a triangle ABC.

i) Centroid of a triangle



Centroid is the point of intersection of medians, whose coordinates are given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

ii) In-centre of a triangle

In-centre is the point of intersection of internal angular bisectors, whose coordinates are given by

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where a, b, c are the lengths of the sides BC, CA, AB respectively.

iii) Ex-centres of a triangle

The point of intersection I_1 of the external angular bisectors of $\angle B$ and $\angle C$ is one of the excentres of the triangle ABC and is given by

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

similarly the other ex-centres are given by

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right) \text{ and}$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

where a, b, c are the lengths of the sides BC, CA, AB respectively.

In some problems we find the Area pretty differently

The area of the triangle formed by the tangent to the curve $y = \frac{8}{4+x^2}$ at $x = 2$ and the coordinate axes is

(*a*) 2 sq. units (*b*) $\frac{7}{2}$ sq. units

(*c*) 4 sq. units (*d*) 8 sq. units.

Solution

(c) From $y = \frac{8}{4 + x^2}$,

when $x = 2, y = \frac{8}{4+4} = 1$

Also, $\frac{dy}{dx} = -\frac{8}{(4+x^2)^2}(2x) \Rightarrow \left[\frac{dy}{dx}\right]_{(2,1)} = -\frac{1}{2}$

$\therefore$ equation of tangent is

$$y - 1 = -\frac{1}{2}(x - 2) \quad \text{or} \quad x + 2y - 4 = 0 \qquad ...(1)$$

Its intercepts on axes are (by putting $y = 0$ and $x = 0$ respectively) $a = 4, b = 2$

$\therefore$ Area $= \frac{1}{2}ab = \frac{1}{2} \times 4 \times 2 = 4$ sq. units.

-

Perpendicular Lines

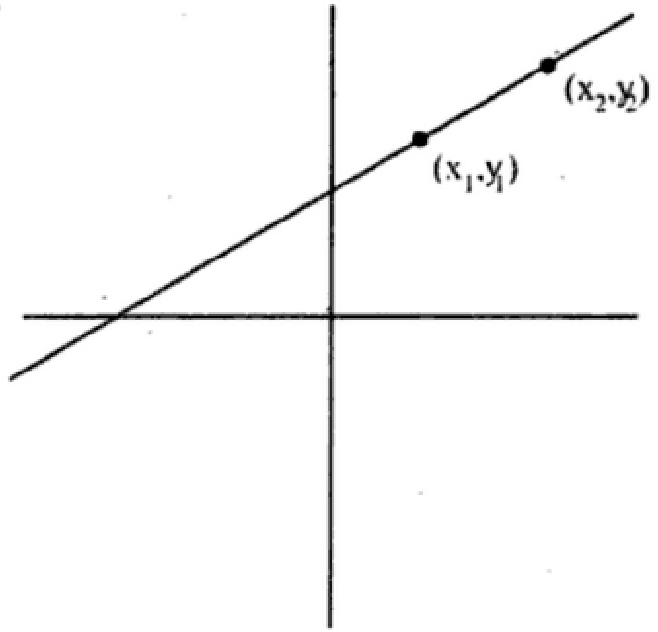
If there is a line whose slope is m (assuming this line NOT parallel to x-axis) then the slope of the line which is perpendicular to this will be -1 / m

Meaning, product of the slopes of lines that are perpendicular is -1

If one of the lines is parallel to x-axis its slope is 0 while the line perpendicular will have a slope of infinity (∞) This line is parallel to y-axis. Product of 0 X ∞ is undefined. In this case we do not apply the -1 as product rule.

-

Equation of the line passing through two points



The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

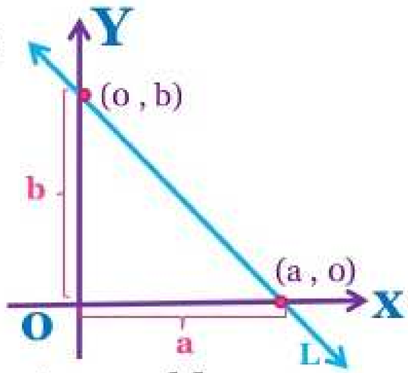
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

-

The intercept form of a line

- Suppose a line **L** makes x-intercept a and y-intercept b on the axes. Obviously **L** meets x-axis at the point (a, o) and y-axis at the point (o, b).

By two-point form of the equation of the line, we have



$$y - 0 = \frac{b-0}{0-a}(x - a)$$

Or $ay = -bx + ab$

i.e., $$\frac{x}{a} + \frac{y}{b} = 1$$

Thus, equation of the line making intercepts **a** and **b** on x- and y-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Question

Through the point $P(\alpha, \beta)$, where $\alpha\beta > 0$ the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area S. If $ab > 0$, then the least value of S is

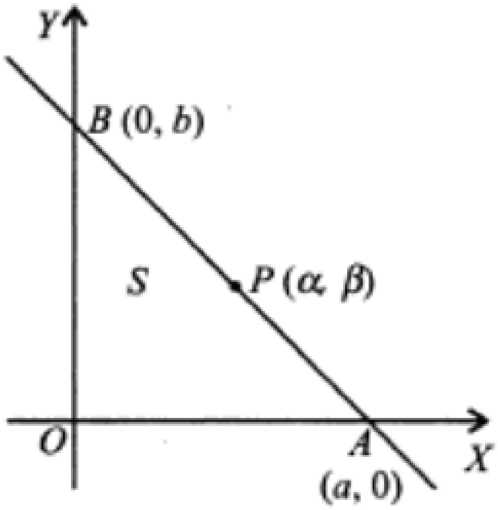
(a) $\alpha\beta$ (b) $2\alpha\beta$

(c) $4\alpha\beta$ (d) none of these

Solution

(b). The equation of the given line is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ...(1)$$



This line cuts x-axis and y-axis at $A\ (a,\ 0)$ and $B\ (0,\ b)$ respectively.

Since area of $\Delta OAB = S$ (Given)

$\therefore \quad \left|\frac{1}{2}ab\right| = S$ or $ab = 2S \quad (\because\ ab > 0) \qquad ...(2)$

Since the line (1) passes through the point $P\ (\alpha,\ \beta)$

$\therefore \quad \frac{\alpha}{a}+\frac{\beta}{b}=1$ or $\frac{\alpha}{a}+\frac{a\beta}{2S}=1 \qquad$ [Using (2)]

or $\quad a^2\beta - 2aS + 2\alpha S = 0.$

Since a is real, $\therefore\ 4S^2 - 8\alpha\beta S \geq 0$

or $\quad 4S^2 \geq 8\alpha\beta S$ or $S \geq 2\alpha\beta \quad \left(\because\ S = \frac{1}{2}ab > 0 \text{ as } ab > 0\right)$

Hence the least value of $\boldsymbol{S = 2\alpha\beta}$.

i) The equation of a line parallel to a given line $ax+by+c=0$ is $ax+by+\lambda=0$, where λ is constant.

ii) The equation of a line perpendicular to a given line $ax+by+c=0$ is $bx-ay+\lambda=0$, where λ is constant.

iii) The slope of the line $ax+by+c=0$ is given by

$$m = \frac{-a}{b}.$$

iv) For intercept on x-axis, put $y=0$. For intercept on y-axis, put $x=0$.

v) Angle θ between the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ is given by

$$\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

vi) The lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ are

a) Coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

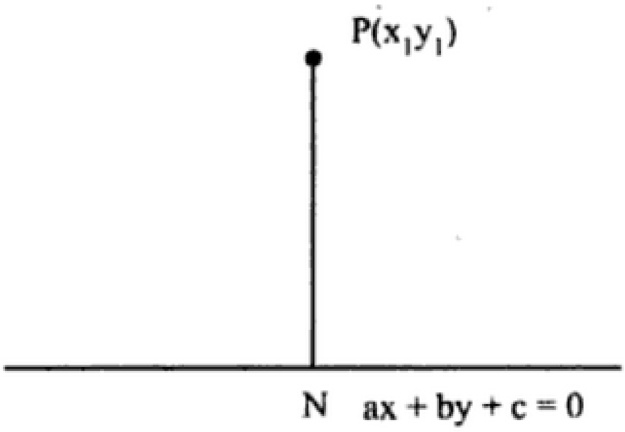
b) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

c) intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

d) Perpendicular if $a_1a_2 + b_1b_2=0$

Distance of a point from a line

The length of the perpendicular from a point (x_1, y_1) to a line $ax+by+c=0$ is given by



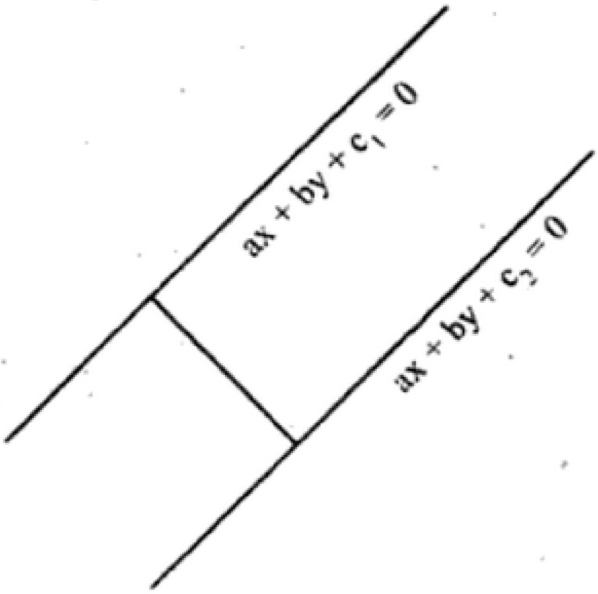
$$PN = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

Note:

The length of the perpendicular from the origin to the line $ax+by+c=0$ is $\frac{|c|}{\sqrt{a^2+b^2}}$

The distance between the parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is given by

$$\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}.$$



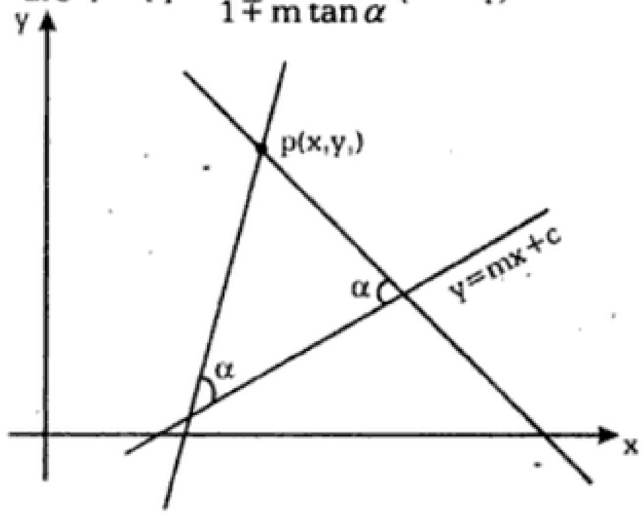
The two points (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax+by+c=0$ according to the quantities ax_1+by_1+c and ax_2+by_2+c have same (or opposite) signs.

The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are *concurrent* (intersect at a point) if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

The equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$

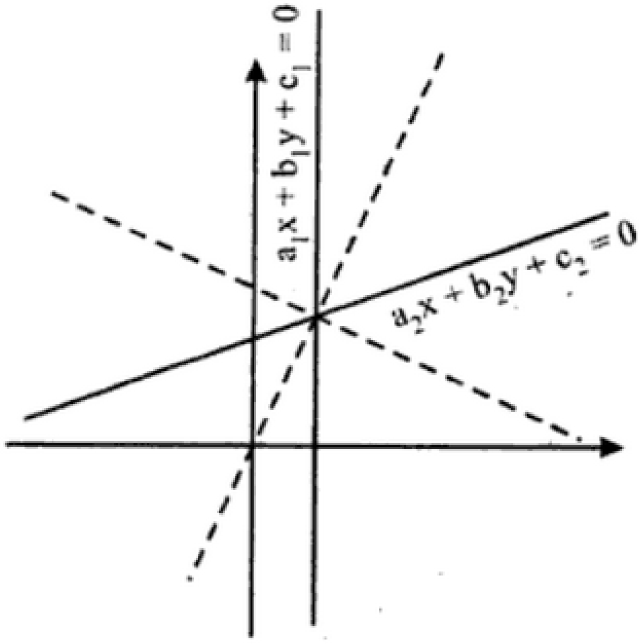
are $y - y_1 = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$



The angle between the lines $x\cos\alpha_1 + y\sin\alpha_1 = P_1$ and $x\cos\alpha_2 + y\sin\alpha_2 = P_2$ is $\alpha_1 - \alpha_2$

Equation of Internal and External bisectors of 2 Lines

The equation of the bisectors of the angles between the lines $a_1x+b_1y+c_1 = 0$ and $a_2x+b_2y+c_2=0$ is given by



$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Bisector of the angle containing the origin

If c_1, c_2 are positive, then the equation of the bisector of the angle containing the origin is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Bisector of Acute and Obtuse angle between lines

i) If c_1, c_2 are positive and if $a_1a_2+b_1b_2>0$, then

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$ is the obtuse angle bisector and

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$ is the acute angle bisector.

ii) If c_1, c_2 are positive and if $a_1a_2+b_1b_2<0$, then

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

is the acute angle bisector and

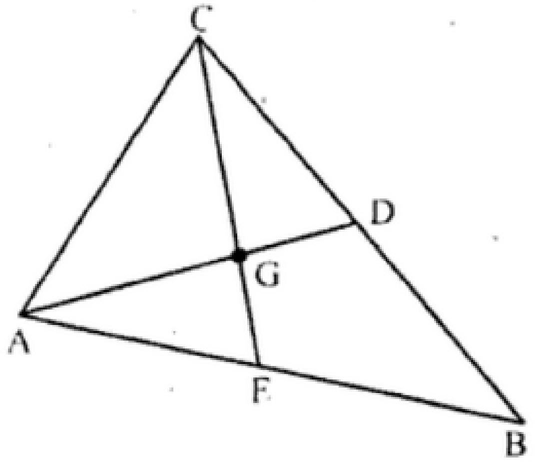
$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

is the obtuse angle bisector.

If c_1, c_2 are positive and $a_1a_2+b_1b_2>0$, then the origin lies in the obtuse angle and the '+' sign gives the bisector of the obtuse angle. If $a_1a_2+b_1b_2<0$, then the origin lies in the acute angle and '+' sign gives the bisector of acute angle.

Coordinates of Centroid, Orthocenter, Circumcenter of a Triangle

Centroid: The point of intersection of the medians of a triangle is called its centroid. It divides the median in the ratio 2:1.



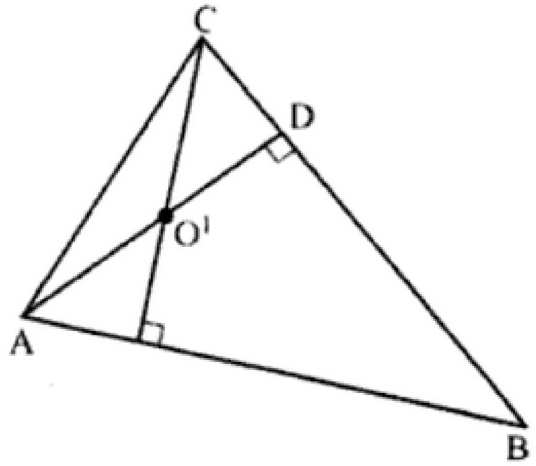
If (x_1, y_1) (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then the coordinates of its centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Orthocentre

The point of intersection of the altitudes of a triangle is called its orthocentre

To determine the orthocentre, first we find equations of line passing through vertices and perpendicular to the opposite sides. Solving two of these three equations we get the co-ordinates of orthocentre.



If angles A, B and C and vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of a ΔABC are given, then orthocentre of ΔABC is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

If any two lines out of three lines, i.e., *AB*, *BC* and *CA* are perpendicular, then orthocentre is the point of intersection of two perpendicular lines.

The orthocentre of the triangle with vertices (0, 0), (x_1, y_1) and (x_2, y_2) is

$$\left\{(y_1 - y_2)\left[\frac{x_1 x_2 - y_1 y_2}{x_2 y_1 - x_1 y_2}\right],\right.$$

$$\left.(x_1 - x_2)\left[\frac{x_1 x_2 + y_1 y_2}{x_1 y_2 - x_2 y_1}\right]\right\}$$

Question on Orthocenter

The orthocentre of the triangle formed by the lines $xy = 0$ and $2x + 3y - 5 = 0$ is

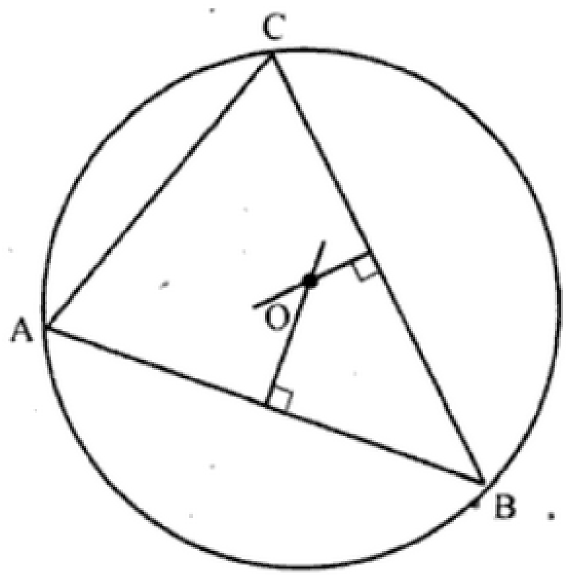
(a) (2, 3) (b) (3, 2) (c) (0, 0) (d) (5, – 5)

Ans. (c)

Solution The given triangle is right angled at (0, 0) which is therefore the orthocentre of the triangle.

Circumcentre

The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circum-centre. It is equidistant from the vertices of a triangle.



Note:

The circumcentre O, centroid G and orthocentre O′ of a triangle ABC are collinear such that G divides O′O in the ratio 2:1

i.e., O′G:OG=2:1

Question

If the circumcentre of a triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$; then the orthocentre lies on the line

(a) $y = (a^2 + 1)x$ (b) $y = 2ax$

(c) $x + y = 0$ (d) $(a-1)^2 x - (a+1)^2 y = 0$

Ans. (d)

Solution We know from geometry that the circumcentre, centroid and orthocentre of a triangle lie on a line. So the orthocentre of the triangle lies on the line joining the circumcentre (0, 0) and the centroid $\left(\frac{(a+1)^2}{2}, \frac{(a-1)^2}{2}\right)$

i.e. $$\frac{(a+1)^2}{2} y = \frac{(a-1)^2}{2} x$$

or $$(a-1)^2 x - (a+1)^2 y = 0.$$

Question

The equations to the sides of a triangle are $x - 3y = 0$, $4x + 3y = 5$ and $3x + y = 0$. The line $3x - 4y = 0$ passes through the

(a) incentre (b) centroid

(c) circumcentre (d) orthocentre of the triangle

Ans. (d)

Solution Two sides $x - 3y = 0$ and $3x + y = 0$ of the triangle being perpendicular to each other, the triangle is right angled at the origin, the point of intersection of these sides. So that origin is the orthocentre of the triangle and the line $3x - 4y = 0$ passes through this orthocentre.

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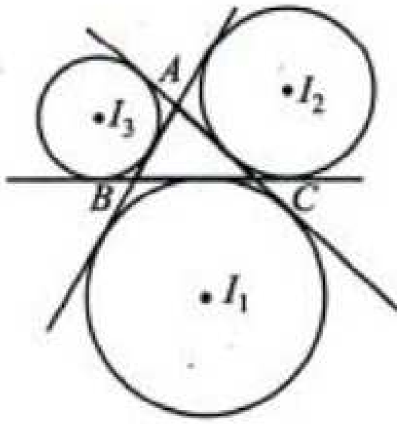
Ex-Centres of a Triangle A circle touches one side outside the triangle and the other two extended sides then circle is known as excircle.



Let ABC be a triangle then there are three excircles, with three excentres I_1, I_2, I_3 opposite to vertices A, B and C respectively. If the vertices of triangle are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ then

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right).$$

Family of lines through the intersection of two given lines

The equation of family of lines passing through the intersection of the lines

$L_1 = a_1x+b_1y+c_1=0$ and $L_2=a_2x+b_2y+c_2=0$ is

$(a_1x+b_1y+c_1) + \lambda (a_2x+b_2y+c_2) = 0$, where λ is a parameter i.e., $L_1+\lambda L_2=0$.

Formulae specific to Pair of Straight Lines

Homogeneous equation of second degree in x and y

A general homogenous equation of degree 2 always represent two straight lines, real or imaginary, through the origin. Conversely, the equal of a pair of lines through origin is a second degree homogeneous equation in x and y.

The equation of the form $ax^2+2hxy+by^2=0$ is called a homogeneous equation of degree 2 in x and y, where a, b, h are constants.

let $ax^2 + 2hxy+by^2=0$ ...(1)

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of Straight lines only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{i.e., iff} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

For easy remembering note that the first row of the Determinant is coeffs of x terms

$(a)x^2 + 2(h)xy$ …. + 2 (g) x …..

Similarly the second row is made of coeffs of y terms. i.e.

2 (h) xy + (b)y^2 + 2 (f) y ….

The last row of the determinant is the last 3 constants of last 3 terms. i.e. g, f, and c

Equation of the lines joining the origin to the points of intersection of a line and a conic.

Let $L \equiv lx + my + n = 0$

and $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

be the equations of a line and a *conic*, respectively. Writing the equation of the line as $\frac{lx+my}{-n} = 1$ and making $S = 0$ homogeneous with its help, we get

$$S = ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

which being a homogeneous equation of second degree, represents a pair of straight lines through the origin and passing through the points common to $S = 0$ and $L = 0$.

-

Equation of the pair of lines through the origin perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Question

If the slope of one of the lines represented by $ax^2 - 6xy + y^2 = 0$ is square of the other, then

(a) $a = 1$ (b) $a = 2$ (c) $a = 4$ (d) $a = 8$

Ans. (d)

Solution Let the lines represented by the given equation be $y = mx$ and $y = m^2x$, then

$$m + m^2 = 6 \text{ and } m^3 = a$$

$\Rightarrow$ $m = 2$ or -3

and so $a = 8$ or -27

So the equation of the required lines is

$$(x + 3y)(3x + y) = 0 \Rightarrow 3x^2 + 10xy + 3y^2 = 0.$$

Question on Locus

If $P(1, 0)$, $Q(-1, 0)$ and $R(2, 0)$ are three given points. The point S satisfies the relation $SQ^2 + SR^2 = 2SP^2$. The locus of S meets PQ at the point

(a) $(0, 0)$ (b) $(2/3, 0)$

(c) $(-3/2, 0)$ (d) $(0, -2/3)$

Ans. (c)

Solution Let S be the point (x, y)

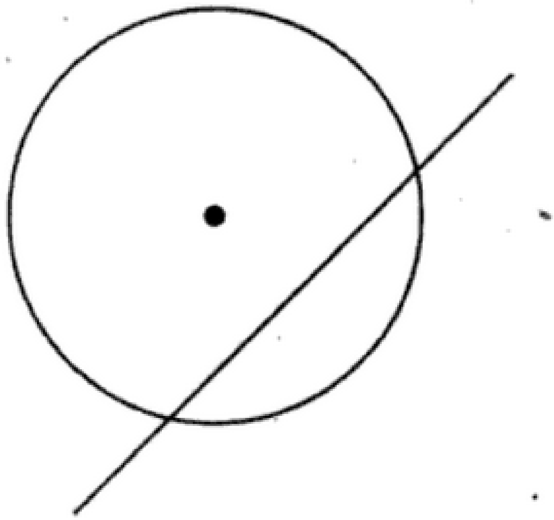
then $(x + 1)^2 + y^2 + (x - 2)^2 + y^2 = 2[(x - 1)^2 + y^2]$

$\Rightarrow 2x + 3 = 0$, the locus of S and equation of PQ is $y = 0$.

So the required points is $(-3/2, 0)$.

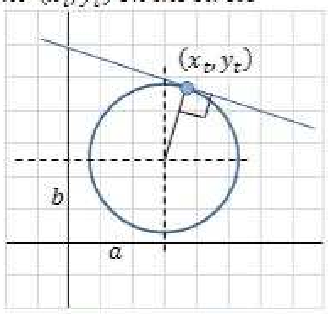
Formulae related to circles

The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ at two distinct points if the length of the perpendicular from the centre is less than the radius of the circle.



ie., $\left|\dfrac{c}{\sqrt{1+m^2}}\right| < a$

Equation of a line tangent to the circle at a point (x_t, y_t) on the circle



Example: *Find the tangent line equation at point (1,2) to the circle*

$$x^2 + y^2 + 2x + 3y - 13 = 0$$

$$m = -\frac{2 \cdot 1 + 2}{2 \cdot 2 + 3} = -\frac{4}{7} = -0.57$$

$$y = -\frac{4}{7}x + \frac{18}{7}$$

and finally $7y = -4x + 18$

Circle form: $(x-a)^2 + (y-b)^2 = r^2$

Tangent line slop is: $m = \dfrac{x_t - a}{b - y_t}$

tangent line equation is: $y = \dfrac{x_t - a}{b - y_t}(x - x_t) + y_t$

provided that $b \neq y_t$

If $b = y_t$ *then the line equation become:* $x = x_t$

Circle form: $x^2 + y^2 + Ax + By + C = 0$

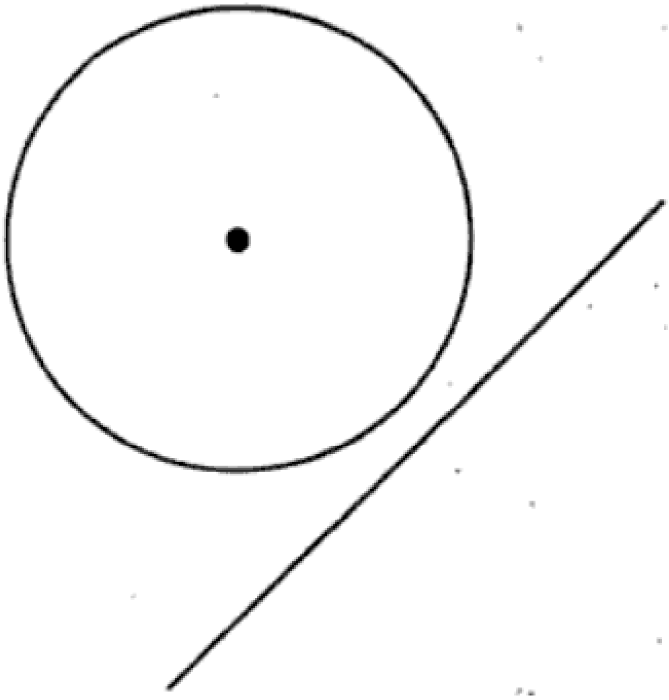
x_t, y_t *should be a point on the circle therfor*

$$x_t^2 + y_t^2 + Ax_t + By_t + C = 0$$

Tangent line slop $m = -\dfrac{2x_t + A}{2y_t + B}$

tangent line equation is: $y = -\dfrac{2x_t + A}{2y_t + B}(x - x_t) + y_t$

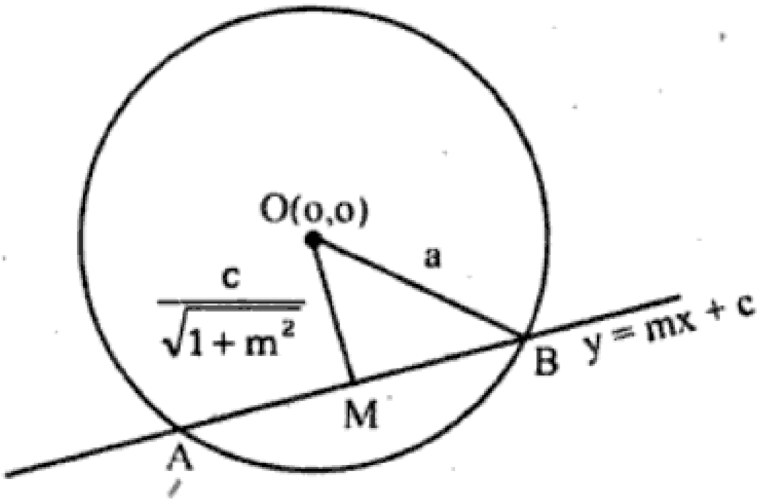
The line does not intersect the circle $x^2 + y^2 = a^2$ if the length of the perpendicular, from the centre is greater than the radius of the circle

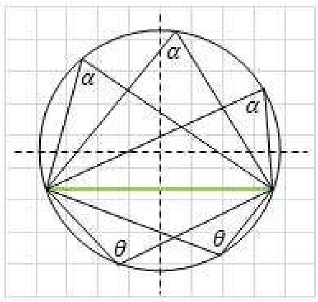


ie., $\left|\dfrac{c}{\sqrt{1+m^2}}\right| > a$

iii) The length of the intercept cut off from a line $y = mx + c$ by a circle $x^2 + y^2 = a^2$ is

- $$2MB = 2\sqrt{\frac{a^2\left(1+m^2\right)-c^2}{\left(1+m^2\right)}}$$



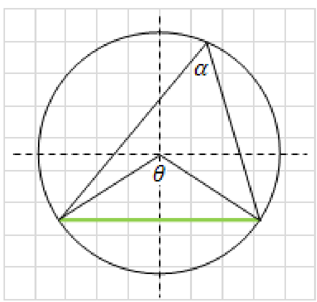


Inscribed angles

(1) All inscribed angles intercepted by the same arc or cord and lies on the same side of the cord are equal.

(2) Sum of opposite angles drawn from the same cord are equal to 180°, $\alpha + \theta = 180°$

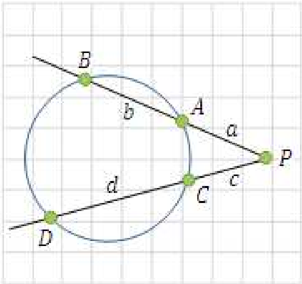
(3) If the cord coincides with the diameter of the circle then the inscribed angle is a right angle $\alpha = \theta = 90°$.



Central and inscribed angles

If central angle θ and inscribed angle α intercepts the same cord or arc then:

$$\theta = 2\alpha \quad or \quad \alpha = \frac{\theta}{2}$$



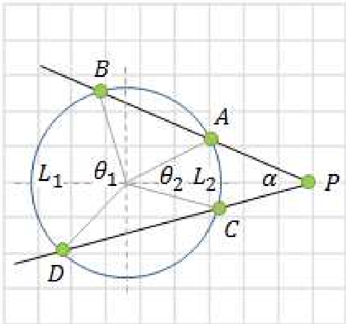
Intersecting secants theorem

If a line from a point P intersects the circle at two different locations then:

$$\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD}$$

We can also write: $(a + b)\,a = (c + d)\,c$

If line PD = c is tangent then $c^2 = (a + b)\,a$

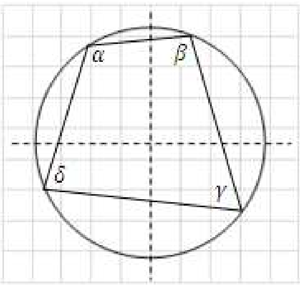


Tangent and secant line theorem

The value of an angle formed by a secant or tangent line drawn from a point P outside the circle equal the half of the difference of the intercepted arces or central angles.

$$\alpha = \frac{1}{2}(L_1 - L_2) = \frac{1}{2}(\theta_1 - \theta_2)$$

θ_1 θ_2 are the central angles of the shown arcs.

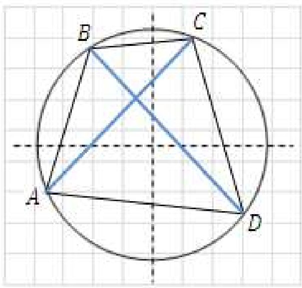


Cyclic quadrilateral

The sum of the opposite angles in an inscribed (cyclic) quadrilateral are equal to 180°.

$$\alpha + \gamma = 180°$$

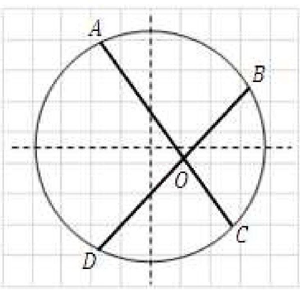
$$\beta + \delta = 180°$$



Ptolemy's theorem

The product of the diagonals of a cyclic quadrilateral equals the sum of the products of the opposite sides.

$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC}$$

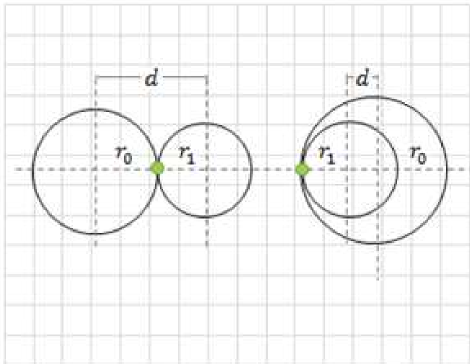


Intersecting chord theorem

The product of the two segments created by intersecting of two chords are equal.

$$\overline{AO} \cdot \overline{OC} = \overline{BO} \cdot \overline{OD}$$

Two circles tangency point (x, y)*: centers of the circles are at* (x_1, y_1) *and* (x_2, y_2)



Example: *Find the tangency point of the circles:* $(x-4)^2 + (y+1)^2 = 9$ *and* $(x+1)^2 + (y+1)^2 = 4$

Check if circles are tangent:

$(-4-1)^2 + (1-1)^2 = (3 \pm 2)^1$

$5 = 5$ *(result with the plus sign)*

so the circles are tangent to each other

$$x = \frac{(-4-1)(9-4)}{2[(1+4)^2 + (1-1)^2]} - \frac{-4+1}{2} = 1$$

$$y = \frac{(1-1)(9-4)}{50} - \frac{1+1}{2} = -1$$

$$(x-a)^2 + (y-b)^2 = r_0^{\,2} \quad and \quad (x-c)^2 + (y-d)^2 = r_1^{\,2}$$

Condition for tangency of two circles:

$$(a-c)^2 + (b-d)^2 = (r_0 \pm r_1)^2$$

(+ sign is for external tangency and − for internal tangency)

If we have the two radii r_0 *and* r_1 *and the distance between the centers* d*, then conditions for tangency are:*

Outer circles tangency: $r_0 + r_1 = d$

Inner circles tangency: $|r_0 - r_1| = d$

Tangency point coordinate:

$$x = \frac{(a-c)(r_0^{\,2} - r_1^{\,2})}{2[(c-a)^2 + (d-b)^2]} - \frac{a+c}{2}$$

$$y = \frac{(b-d)(r_0^{\,2} - r_1^{\,2})}{2[(c-a)^2 + (d-b)^2]} - \frac{b+d}{2}$$

$$x^2 + y^2 + Ax + By + C = 0 \quad and \quad x^2 + y^2 + Dx + Ey + F = 0$$

Condition for tangency of two circles:

$$(A-D)^2 + (B-E)^2 = \left(\sqrt{A^2 + B^2 - 4C} \pm \sqrt{D^2 + E^2 - 4F}\right)^2$$

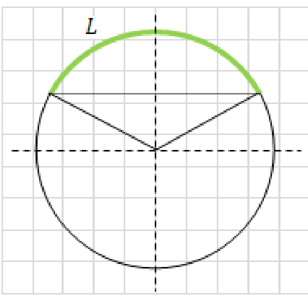
Tangency point coordinate:

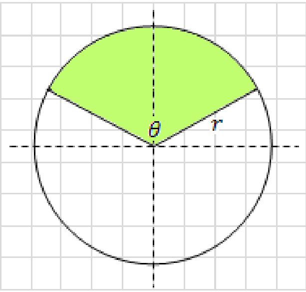
Substitute $u = F - C \quad v = D - A \quad w = B - E$

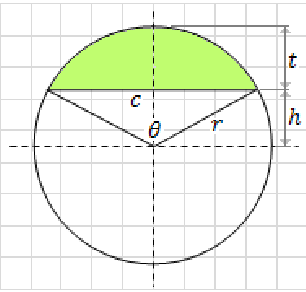
$$x = \frac{2uv + Aw^2 + Buw}{2(v^2 + w^2)}$$

$$y = \frac{-2uw + Bv^2 + Avw}{2(v^2 + w^2)}$$

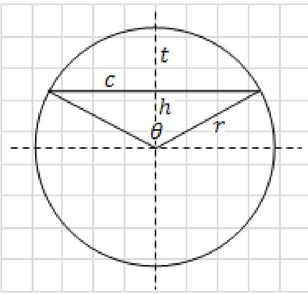
Arc Segment and Sector

Circle arc	*(θ in radians)*	*(θ in degree)*
	$L = Arc\ length$ $L(r, \theta) = r\theta$ $L(r, c) = 2r \sin^{-1}\left(\frac{c}{2r}\right)$ $L(r, t) = 2r \cos^{-1}\left(\frac{r-t}{r}\right)$	$L = r\frac{\theta}{180}\pi$

$Circle\ sector$	$(\theta\ in\ radians)$	$(\theta\ in\ degree)$
	$A(r,\theta) = \frac{\theta}{2}r^2$ $A(r,L) = \frac{1}{2}Lr$ $A(r,c) = r^2 \sin^{-1}\left(\frac{c}{2r}\right)$	$A = \frac{\theta}{360}\pi r^2$
	$P(r,\theta) = r(\theta + 2)$ $P(r,c) = 2r\left(\sin^{-1}\left(\frac{c}{2r}\right) + 1\right)$	$P = r\left(\frac{\theta}{180}\pi + 2\right)$

$Circle\ segment$	$(\theta\ in\ radians)$	$(\theta\ in\ degrees)$
	$A(r,\theta) = \frac{r^2}{2}(\theta - \sin\theta)$ $A(r,c) = r^2 \sin^{-1}\left(\frac{c}{2r}\right) - \frac{c}{4}\sqrt{4r^2 - c^2}$ $A(r,h) = r^2 \cos^{-1}\left(\frac{h}{r}\right) - h\sqrt{r^2 - h^2}$ $A(r,t) = r^2 \cos^{-1}\left(\frac{r-t}{r}\right) - (r-t)\sqrt{r^2 - 2rt}$	$A = \frac{r^2}{2}\left(\frac{\pi}{180}\theta - \sin\theta\right)$
	$P(r,\theta) = r\left(\theta + 2\sin\frac{\theta}{2}\right)$ $P(r,c) = r^2 \sin^{-1}\left(\frac{c}{2r}\right) - \frac{c}{2}\sqrt{4r^2 - c^2}$	$P = r\left(\frac{\pi}{180}\theta + 2\sin\frac{\theta}{2}\right)$

$Segment\ sector\ realations$



$area\ (A),\ circumference\ (P)$

$$A_{segment} = A_{sector} - A_{triangle}$$

	$f(r,\theta)$	$f(r,h)$	$f(r,t)$	$f(r,c)$	$f(h,c)$	$f(h,t)$
$c =$	$2r\sin\frac{\theta}{2}$	$2\sqrt{r^2 - h^2}$	$2\sqrt{2rt - t^2}$	–	–	$2\sqrt{t^2 + 2th}$
$t =$	$r\left(1 - \cos\frac{\theta}{2}\right)$	$r - h$	–	$r - \frac{1}{2}\sqrt{4r^2 - c^2}$	$\frac{1}{2}\sqrt{4h^2 + c^2} - h$	–
$h =$	$r\cos\frac{\theta}{2}$	–	$r - t$	$\frac{1}{2}\sqrt{4r^2 - c^2}$	–	–
$r =$	–	–	–	–	$\frac{1}{2}\sqrt{4h^2 + c^2}$	$h + t$
$\theta =$	–	$2\cos^{-1}\left(\frac{h}{r}\right)$	$2\cos^{-1}\left(\frac{r-t}{r}\right)$	$2\sin^{-1}\left(\frac{c}{2r}\right)$	$2\tan^{-1}\left(\frac{c}{2h}\right)$	$2\cos^{-1}\left(\frac{h}{h+t}\right)$
$A_t =$	$\frac{r^2}{2}\sin\theta$	$h\sqrt{r^2 - h^2}$	$(r-t)\sqrt{2rt - t^2}$	$\frac{c}{4}\sqrt{4r^2 - c^2}$	$\frac{1}{2}ch$	$h\sqrt{t^2 + 2th}$
$P_t =$	$r\left(2\sin\frac{\theta}{2} + 2\right)$	$2\left(r + \sqrt{r^2 - h^2}\right)$	$2\left(r + \sqrt{2rt - t^2}\right)$	$2r + c$	$c + \sqrt{4h^2 + c^2}$	$2\left(h + t + \sqrt{t^2 + 2th}\right)$

	$f(h,\theta)$	$f(c,\theta)$	$f(c,t)$	$f(\theta,t)$
$c =$	$2h\tan\frac{\theta}{2}$	–	–	$\frac{2t\sin\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}$
$t =$	$h\left(\frac{1}{\cos\frac{\theta}{2}}-1\right)$	$\frac{c}{2}\left(\frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)$	–	–
$h =$	–	$\frac{c}{2\tan\frac{\theta}{2}}$	$\frac{c^2-4t^2}{8t}$	$\frac{t\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}$
$r =$	$\frac{h}{\cos\frac{\theta}{2}}$	$\frac{c}{2\sin\frac{\theta}{2}}$	$\frac{4t^2+c^2}{8t}$	$\frac{t}{1-\cos\frac{\theta}{2}}$
$\theta =$	–	–	$\sin^{-1}\left(\frac{4ct}{4t^2+c^2}\right)$	–
$A_t =$	$h^2\tan\frac{\theta}{2}$	$\frac{c^2}{4\tan\frac{\theta}{2}}$	$\frac{c}{16t}(c^2-4t^2)$	$\frac{t^2\sin\theta}{2\sin^2\frac{\theta}{2}}$
$P_t =$	$2h\left(\frac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$	$c\left(\frac{1}{\sin\frac{\theta}{2}}+1\right)$	$c+\frac{4t^2+c^2}{4t}$	$\frac{2t\left(1+\sin\frac{\theta}{2}\right)}{1-\cos\frac{\theta}{2}}$

Question on Tangent

The point on the curve $y = 6x - x^2$ where the tangent is parallel to x-axis is

(*a*) (0, 0) (*b*) (2, 8)

(*c*) (6, 0) (*d*) (3, 9).

Solution

(*d*) $\frac{dy}{dx} = 6 - 2x$

$\therefore \frac{dy}{dx} = 0 \Rightarrow x = 3.$

$\therefore y = 18 - 9 = 9 \quad \therefore$ Point is (3, 9).

Question

For the curve $x = t^2 - 1$, $y = t^2 - t$, the tangent line is perpendicular to x- axis, where

(a) $t = 0$ (b) $t \to \infty$

(c) $t = \frac{1}{\sqrt{3}}$ (d) $t = -\frac{1}{\sqrt{3}}$.

Solution

(a) $\frac{dx}{dt} = 2t$,

Tangent is perpendicular to x-axis if $\frac{dx}{dt} = 0 \Rightarrow t = 0$.

Question

The point on the curve $y^2 = x$, the tangent at which makes an angle of 45° with x-axis will be given by

(a) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) (2, 4) (d) $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Solution

(d) $y^2 = x \Rightarrow 2y\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y} = \tan 45° = 1$ (given)

$\Rightarrow y = \frac{1}{2}$. $\therefore x = \frac{1}{4}$

$\therefore$ Point is $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Question

If tangent to the curve $x = at^2, y = 2at$ is perpendicular to x-axis then its point of contact is

(a) (a, a) (b) $(0, a)$

(c) $(a, 0)$ (d) $(0, 0)$.

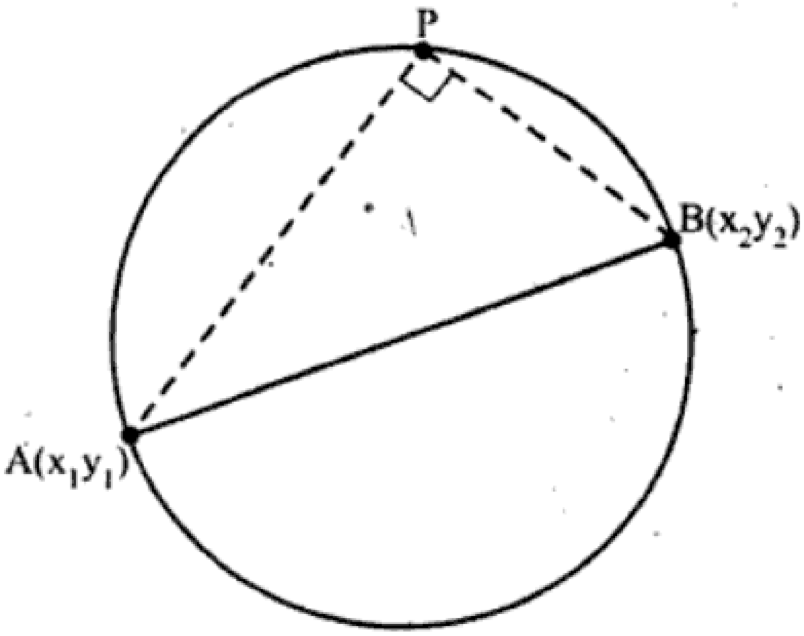
Solution

(d) $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a \Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow \frac{1}{t} = \infty \Rightarrow t = 0 \Rightarrow$ Point is $(0, 0)$.

Equation of the circle when the end points of a diameter are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of circle and let P be any point on circle.



Now, since the angle subtended at the point P in the semicircle APB is a right angle.

$m_1 m_2 = -1$ (m_1 = slope of AP, m_2 = slope of BP)

$$\frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

ie., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Condition for two intersecting circles to be orthogonal

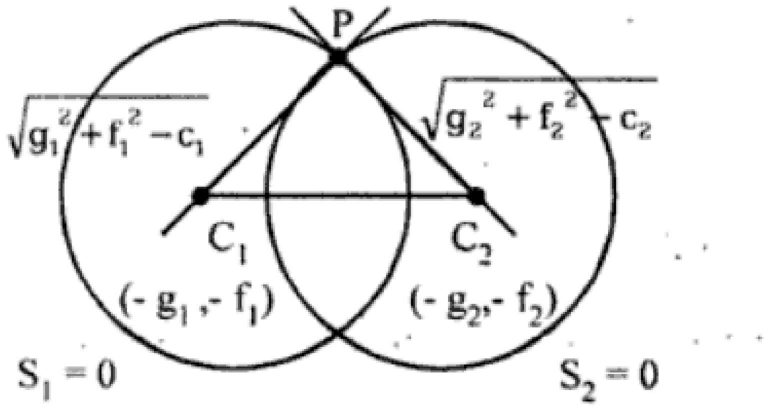
Definition

Two intersecting circles are said to cut each other orthogonally when the tangents at the point of intersection of the two circles are at right angles.

Let the circles

$S_1 \equiv x^2 + y^2 + 2g_1 x + 2f_1 y + C_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2 x + 2f_2 y + C_2 = 0$



intersect orthogonally, then $\angle C_1PC_2 = 90°$

ie., ΔC_1PC_2 is right angled

$\therefore C_1C_2^2 = C_1P^2 + C_2P^2$

$(g_1 - g_2)^2 + (f_1 - f_2)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2)$

$\Rightarrow 2g_1g_2 + 2f_1f_2 = c_1 + c_2$ is the required condition that S_1 and S_2 intersect orthogonally.

Some important results

i) The equation of chord joining two points θ_1 and θ_2 on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$(x + g)\cos\frac{\theta_1+\theta_2}{2} + (y + f)\sin\frac{\theta_1+\theta_2}{2} = r\cos\left(\frac{\theta_1-\theta_2}{2}\right),$$ where r is the radius of the circle.

ii) The equation of the tangent at $P(\theta)$ on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x+ g)\cos\theta + (y + f)\sin\theta = \sqrt{g^2 + f^2 - c}$

iii) The locus of the point of intersection of two tangents drawn to the circle $x^2 + y^2 = a^2$ which makes an constant angle α to each other is $x^2 + y^2 - 2a^2 = 4a^2(x^2 + y^2 - a^2)\cot^2\alpha$.

Question

The equation of tangent to the circle $x^2 + y^2 + 6x + 4y - 12 = 0$ at (6,2) is

a) $4x - 9y - 6 = 0$ **b)** $9x + 4y + 12 = 0$

b) $3x - 9y = 0$ **d)** $2x - 3y = 6$

Ans (b)

Note:

The equation of tangent at (x_1, y_1) is

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

thus the equation of tangent at (6,2) is

$6x + 2y + 3(x+6) + 2(y+2) - 12 = 0$

i.e., $9x + 4y + 12 = 0$.

Question on Angle of intersection

The angle of intersection of the curves $y = x^2$ and $6y = 7 - x^3$ at (1, 1) is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) None of these.

Solution

(c) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow m_1 = 2$

$6y = 7 - x^3 \Rightarrow \frac{dy}{dx} = -\frac{1}{2} \Rightarrow m_2 = -\frac{1}{2}$

$\therefore \quad m_1 m_2 = -1$ at (1, 1)

$\Rightarrow \quad \theta = \frac{\pi}{2}$.

Question

If a, x_1, x_2 are in G.P. with common ratio r, and b, y_1, y_2 are in G.P. with common ratio s where $s - r = 2$, then the area of the triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2) is

(a) $|ab(r^2 - 1)|$ (b) $ab(r^2 - s^2)$

(c) $ab(s^2 - 1)$ (d) $abrs$

Ans. (a)

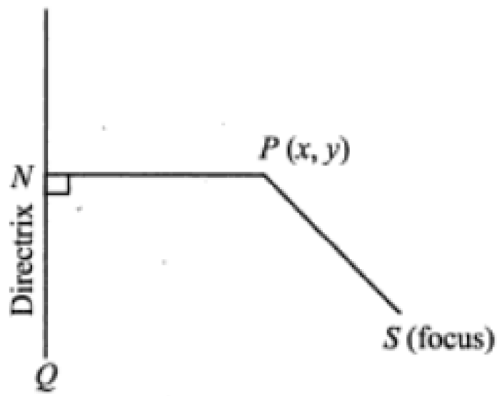
Solution Area of the triangle

$$= \frac{1}{2}\left|\begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix}\right| = \frac{1}{2}|ab(r-1)(s-1)(s-r)|$$

$$= |ab(r-1)(r+1)| = |ab(r^2-1)|$$

ELLIPSE

An ellipse is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point (called focus) and a fixed line (called directrix) is a constant which is less than one. This ratio is called eccentricity and is denoted by e. For an ellipse, $e < 1$.



Let S be the focus, QN be the directrix and P be any point on the ellipse. Then, by definition, $\frac{PS}{PN} = e$ or $PS = e\,PN$, $e < 1$, where PN is the length of the perpendicular from P on the directrix QN.

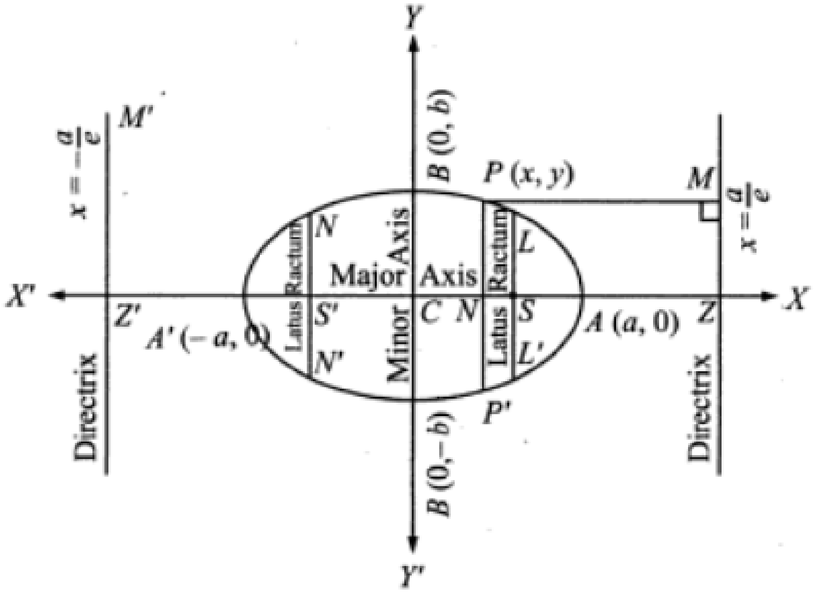
An alternate definition An ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant.

EQUATION OF AN ELLIPSE IN STANDARD FORM

The standard form of the equation of an ellipse is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ (a>b),$$

where a and b are constants.



SOME TERMS AND PROPERTIES RELATED TO AN ELLIPSE

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$, has been shown in the figure given above.

1. Symmetry

(a) On replacing y by $-y$, the above equation remains unchanged. So, the curve is symmetrical about x-axis

(b) On replacing x by $-x$, the above equation remains unchanged. So, the curve is symmetrical about y-axis

2. Foci If S and S' are the two foci of the ellipse and their coordinates are (ae, o) and $(-ae, 0)$ respectively, then distance between foci is given by

$$SS' = 2ae.$$

3. Directrices If ZM and $Z'M'$ are the two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance between directrices is given by

$$ZZ' = \frac{2a}{e}.$$

4. Axes The lines AA' and BB' are called the **major axis** and **minor axis** respectively of the ellipse.

The length of major axis = $AA' = 2a$

The length of minor axis = $BB' = 2b$

5. Centre The point of intersection C of the axes of the ellipse is called the centre of the ellipse. All chords, passing through C are bisected at C.

6. Vertices The end points A and A' of the major axis are known as the vertices of the ellipse

$$A \equiv (a, 0) \text{ and } A' \equiv (-a, 0)$$

Remember: The vertex divides the join of focus and the point of interesection of directrix with axis internally and enternally the ratio $e : 1$.

7. Focal chord A chord of the ellipse passing through its focus is called a focal chord.

8. Ordinate and double ordinate Let P be a point on the ellipse. From P, draw $PN \perp AA'$ (major axis of the ellipse)and produce PN to meet the ellipse at P'. Then PN is called an *ordinate* and PNP' is called the *double ordinate* of the point P.

9. Latus rectum If LL' and NN' are the latus rectum of the ellipse, then these lines are $\perp$ to the major axis AA', passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \quad L' \equiv \left(ae, \frac{-b^2}{a}\right),$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = NN'$.

10. By definition, $SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$

and $\quad S'P = e\left(\frac{a}{e} + x\right) = a + ex.$

This implies that distances of any point $P(x, y)$ lying on the ellipse from foci are : $(a - ex)$ and $(a + ex)$. In other words

$$SP + S'P = 2a$$

i.e., sum of distances of any point P (x, y) lying on the ellipse from foci is constant.

11. Eccentricity of the ellipse Since, $SP = ePM$, therefore,

$$SP^2 = e^2PM^2$$

or
$$(x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$(x - ae)^2 + y^2 = (a - ex)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

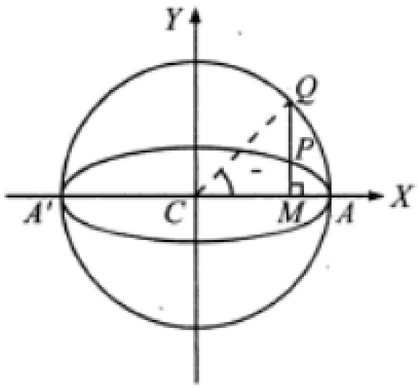
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

On comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$b^2 = a^2(1 - e^2) \quad \text{or} \quad e = \sqrt{1 - \frac{b^2}{a^2}}.$$

12. Auxiliary circle The circle drawn on major axis AA' as diameter is known as the Auxiliary circle.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then the equation of its auxiliary circle is:



$$x^2 + y^2 = a^2.$$

Let Q be a point on auxiliary circle so that QM, perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding points on the ellipse and auxiliary circle respectively.

The angle θ is known as *eccentric angle* of the point P on the ellipse.

It may be noted that the CQ and not CP is inclined at θ with x-axis.

13. Parametric equation of the ellipse The coordinates $x = a\cos\theta$ and $y = b\sin\theta$ satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for all real values of θ. Thus, $x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.

Hence the coordinates of any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may be taken as $(a\cos\theta, b\sin\theta)$. This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point $(a\cos\theta, b\sin\theta)$ on the ellipse.

14. Equation of Chord The equation of the chord joining the points $P \equiv (a\cos\theta_1, b\sin\theta_1)$ and $Q \equiv (a\cos\theta_2, b\sin\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

Remember: If the centre of the ellipse lies at (h, k) and the axes are parallel to the coordintate axes, then the equation of the ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point $P(x_1, \ y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = 0$ or < 0.

Intersection of line and an Ellipse

The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$ and does not intersect if $a^2m^2 + b^2 < c^2$.

CONDITION FOR TANGENCY AND POINTS OF CONTACT

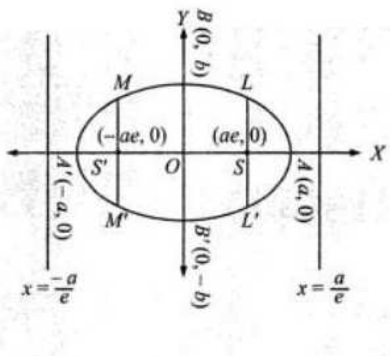
The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are

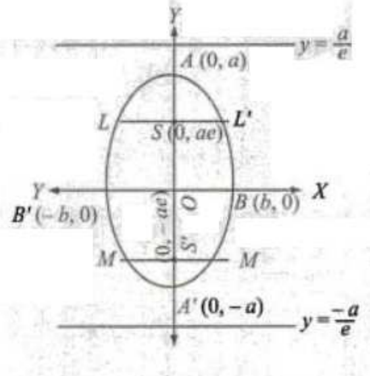
$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Two standard forms of the ellipse

standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ (Horizontal Form of an Ellipse)	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1\ (a, b)$ (Vertical Form of an Ellipse)

Shape of the Ellipse



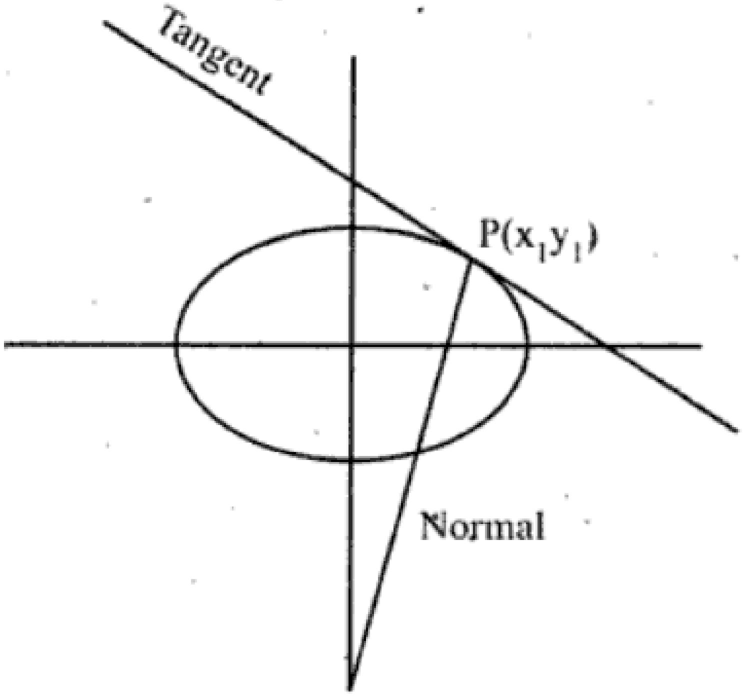


Centre	$(0, 0)$	$(0, 0)$
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Equation of directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
Eccentricity	$e = \sqrt{\frac{a^2-b^2}{a^2}}$	$e = \sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Ends of latra-recta	$\left(\pm ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{b^2}{a}, \pm ae\right)$
Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$

Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = a - ey_1$ and $S'P = a + ey_1$.
Sum of focal radii $SP + S'P =$	$2a$	$2a$
Distance between foci	$2ae$	$2ae$
Distance between directrices	$\frac{2a}{e}$	$\frac{2a}{e}$
Tangents at the vertices	$x = \pm a$	$y = \pm a$

Formulae related to ellipse

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$



The equation of normal to the ellipse

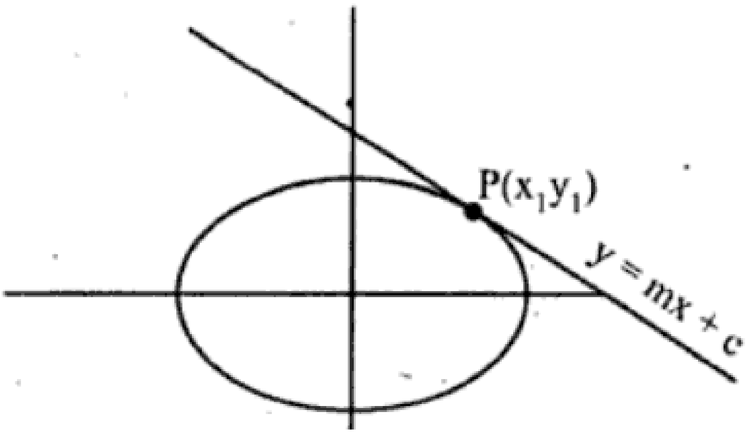
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1y_1)$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Note:

Four normals can be drawn from any point to the ellipse.

Condition for y = mx + c to be a tangent to the ellipse and points of tangency

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1 y_1)$ is



$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad ...(1)$$

Given $-mx + y = c$ ..(2)

(1) and (2) represent the same line

thus $\dfrac{\frac{x_1}{a^2}}{-m} = \dfrac{\frac{y_1}{b^2}}{1} = \dfrac{1}{c}$

$$\Rightarrow x_1 = \frac{-a^2m}{c}, \; y_1 = \frac{b^2}{c}$$

Since $P(x_1, y_1)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

we get, $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$

$$\Rightarrow \frac{a^4m^2}{c^2a^2} + \frac{b^4}{c^2b^2} = 1$$

CHORD WITH A GIVEN MID POINT

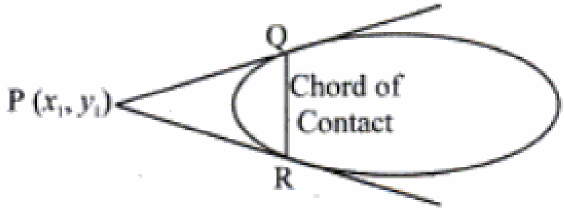
The equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1$$

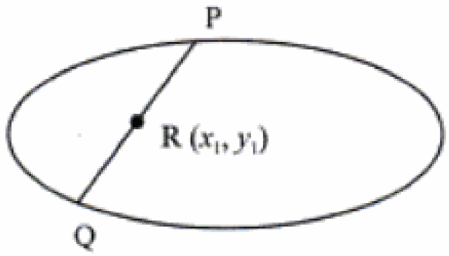
where $T \equiv \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$ and $S_1 \equiv \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1.$

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $T = 0$, where



$$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1.$$



So Review the formulae

The following are some standard results for an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

1. The parametric equations of an ellipse (hyperbola) or the coordinates of any point on the ellipse (hyperbola) are $x = a \cos \theta, y = b \sin \theta$ ($x = a \sec \theta$, $y = b \tan \theta$). The point is denoted "θ".
2. An equation of the tangent at the above point "θ" is
$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \ \left(\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1\right)$$
3. An equation of the normal at the same point "θ" is
$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \ \left(\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2\right)$$

4. An equation of the tangent at the point $P(x', y')$ on the ellipse is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, results corresponding to (4) – (6) and (8) are obtained by replacing b^2 by $(-b^2)$.

5. The condition that the line $y = mx + c$ touches the ellipse is $c^2 = a^2m^2 + b^2$, so that the equation of any tangent to the ellipse (not parallel to the y-axis) can be written as $y = mx \pm \sqrt{a^2m^2 + b^2}$.
6. *Director circle* of an ellipse is the locus of the point of intersection of tangents to the ellipse which intersect at right angles and its equation is $x^2 + y^2 = a^2 + b^2$.
7. *Auxiliary circle* of an ellipse is the circle on major axis of the ellipse as diameter and its equation is $x^2 + y^2 = a^2$.
   If P is a point on the ellipse and Q is a point on the auxiliary circle such that Q lies on the ordinate produced of the point P, then $\angle ACQ$ (where CA is the semimajor axis of the ellipse) is called the *eccentric angle* of the point P on the ellipse and the coordinates of P are $(a \cos \phi, b \sin \phi)$ where $\phi = \angle ACQ$.

8. A diameter of an ellipse is the locus of the mid points of a system of parallel chords of the ellipse and its equation is

$$y = -\frac{b^2}{a^2 m} x,$$

where m is the slope of the parallel chords of the ellipse which are bisected by it. This is a line through the centre of the ellipse. Two diameters of an ellipse are said to be *conjugate* when each bisects the chords parallel to the others. Thus two diameters $y = m x$ and $y = m'x$ of the ellipse are conjugate if

$$m m' = -\frac{b^2}{a^2}.$$

9.

A hyperbola whose asymptotes are perpendicular to each other is called a *rectangular hyperbola* and its equation is $x^2 - y^2 = a^2$. By taking the asymptotes of the rectangular hyperbola as the coordinate axes, its equation can be written as $x y = c^2$ (where $c^2 = a^2/2$) and the parametric equation of this rectangular hyperbola is $x = c t$, $y = c/t$, t being the parameter.

An asymptote to a curve is a line which touches the curve at infinity. Thus equation of the asymptotic of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Question

The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is

(a) 0 (b) 1 (c) 2 (d) infinite

Ans. (c)

Solution We know that $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2 m^2 + b^2$

Here $m = a^2 = 4, b^2 = 1$ so $c^2 = 4 \times 4^2 + 1 \quad \Rightarrow \quad c = \pm\sqrt{65}$

Question

The focii of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

(a) 5 (b) 7 (c) 9 (d) 1

Ans. (b)

Solution $16 - b^2 = \frac{144}{25} + \frac{81}{25} \quad \Rightarrow \quad b^2 = 7.$

Question

The normal to the curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is

(a) ellipse (b) parabola

(c) circle (d) hyperbola or ellipse

Ans. (d)

Solution Equation of the normal at (x, y) is $Y - y = -\frac{dx}{dy}(X - x)$ which meets the x-axis at $G\left(x + y\frac{dy}{dx}\right)$, then $x + y\frac{dy}{dx} = \pm 2x$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 2x \quad \Rightarrow \quad y\,dy = x\,dx \quad \Rightarrow \quad x^2 - y^2 = c$$

or $$y\,dy = -3x\,dx \quad \Rightarrow \quad 3x^2 + y^2 = c$$

Thus the curve is either hyperbola or ellipse.

Formulae related to Hyperbola

Parametric equations of the hyperbola

A point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be represented as $x = a \sec\theta$, $y = b \tan\theta$ in a single parameter θ. These equations are called parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The point $(a \sec\theta, b \tan\theta)$ is simply denoted by θ.

Some important results

i) The equation of the chord joining the points $(a \sec\alpha, b \tan\alpha)$ and $(a \sec\beta, b \tan\beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}.$$

ii) The equation of the tangent at $P(\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

iii) The equation of the normal at $P(\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

iv) The condition that the line $lx + my + n = 0$ may be a normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

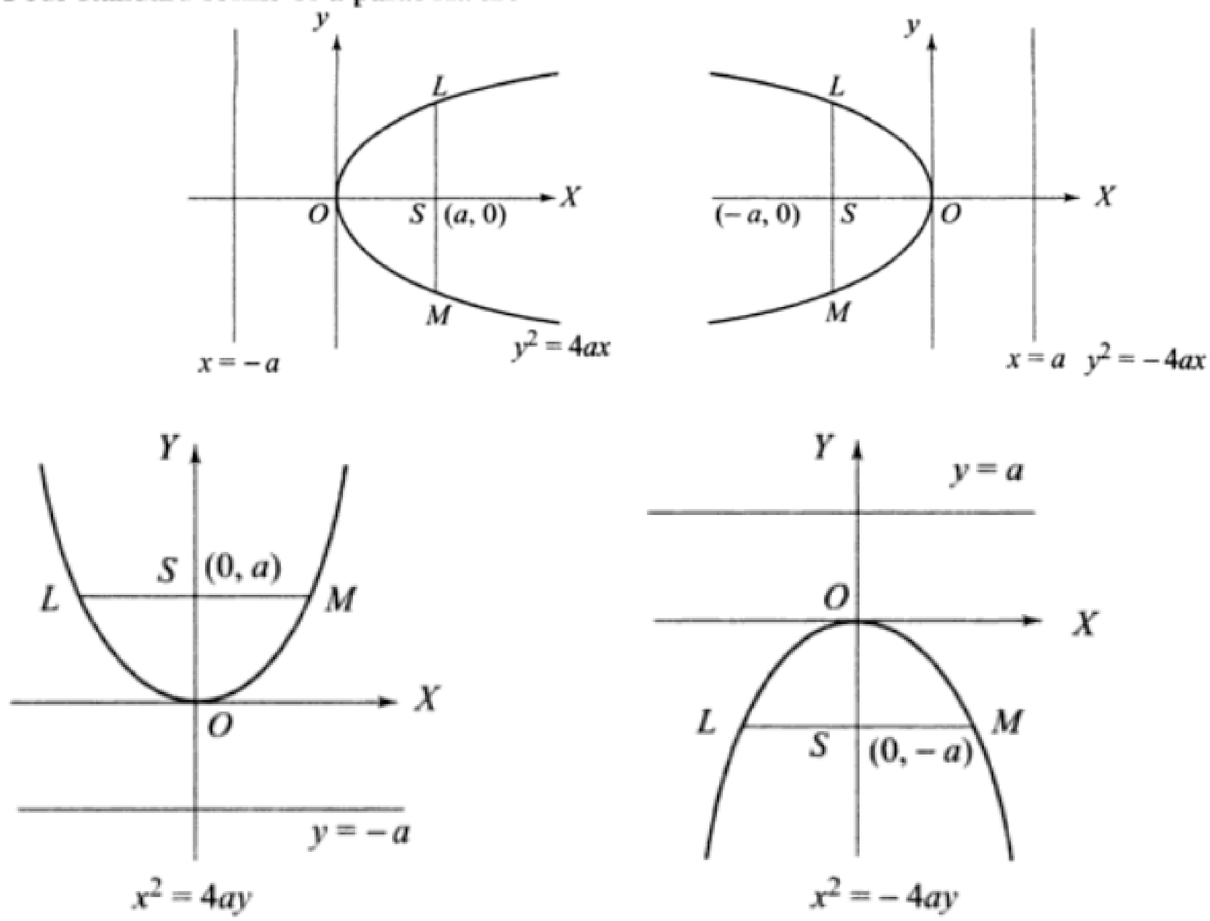
v) If P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci S and S^1, then $S^1P - SP = 2a$.

vi) The locus of point of intersection of perpendicular tangents to an hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a circle $x^2 + y^2 = a^2 - b^2$ called director circle of the hyperbola.

vii) The locus of the feet of perpendiculars drawn the foci to any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a circle $x^2 + y^2 = a^2$, called auxiliary circle of the hyperbola.

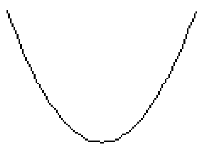
Parabola

$y^2 = 4ax$ is a *standard form* of the equation of a parabola.
Four standard forms of a parabola are



PARABOLAS HAVE TWO SHAPES: CONCAVE UP OR CONCAVE DOWN

concave-up parabola: "a" is positive in $ax^2 + bx + c$



Examples of concave up equations

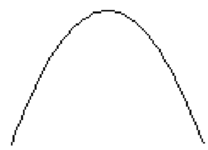
$y = 3x^2 + 2x + 1$

$y = x^2 - 3x$

$y = 6x^2 + 2$

$y = 5x^2$

concave-down parabola: "a" is negative in $ax^2 + bx + c$



Examples of concave down equations

$y = -3x^2 + 2x + 1$

$y = -1x^2 - 3x$

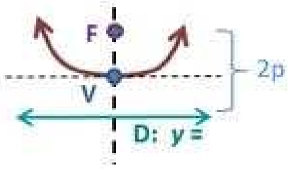
$y = -6x^2 + 2$

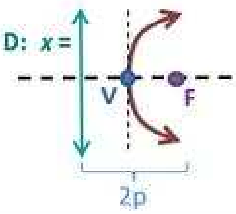
$y = -5x^2$

The following terms are used in context of the parabola $y^2 = 4ax$.

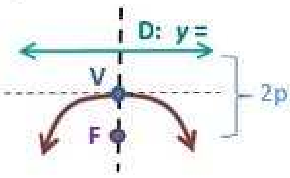
1. The point $O(0, 0)$ is the *vertex* of the parabola, and the tangent to the parabola at the vertex is $x = 0$.
2. The line joining the vertex O and the focus $S(a, 0)$ is the *axis* of *the parabola* and its equation is therefore $y = 0$.
3. Any chord of the parabola perpendicular to its axis is called a *double ordinate*.
4. Any chord of the parabola passing through its focus is called a *focal chord.*
5. The focal chord of the parabola perpendicular to its axis is called *its latus rectum*; the length of this latus rectum is therefore $4a$.
6. The points on a parabola, the normals at which are concurrent, are called *co-normal points* of the parabola. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are conormal points of the parabola $y^2 = 4ax$, then $y_1 + y_2 + y_3 = 0$.
7. A line which bisects a system of parallel chords of a parabola is called a *diameter* of the parabola.

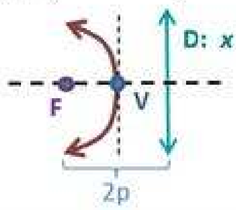
Vertical Parabola	Horizontal Parabola
Positive Coefficient At $(0,0)$: $y = ax^2$ **General:** $y = a(x-h)^2 + k$ or $y - k = a(x-h)^2$ $y = \frac{1}{4p}(x-h)^2 + k$ or $y - k = \frac{1}{4p}(x-h)^2$ or $4p(y-k) = (x-h)^2$ **Vertex:** (h,k) **Axis of Symmetry:** $x = h$	**Positive Coefficient** At $(0,0)$: $x = ay^2$ **General:** $x = a(y-k)^2 + h$ or $x - h = a(y-k)^2$ $x = \frac{1}{4p}(y-k)^2 + h$ or $x - h = \frac{1}{4p}(y-k)^2$ or $4p(x-h) = (y-k)^2$ **Vertex:** (h,k) **Axis of Symmetry:** $y = k$





Vertical Parabola	Horizontal Parabola
Negative Coefficient At $(0,0)$: $y = -ax^2$ **General:** $y = -a(x-h)^2 + k$ or $y - k = -a(x-h)^2$ $y = -\frac{1}{4p}(x-h)^2 + k$ or $y - k = -\frac{1}{4p}(x-h)^2$ or $-4p(y-k) = (x-h)^2$ **Vertex:** (h,k) **Axis of Symmetry:** $x = h$	**Negative Coefficient** At $(0,0)$: $x = -ay^2$ **General:** $x = -a(y-k)^2 + h$ or $x - h = -a(y-k)^2$ $x = -\frac{1}{4p}(y-k)^2 + h$ or $x - h = -\frac{1}{4p}(y-k)^2$ or $-4p(x-h) = (y-k)^2$ **Vertex:** (h,k) **Axis of Symmetry:** $y = k$





The following are some *standard results for the parabola* $y^2 = 4ax$:

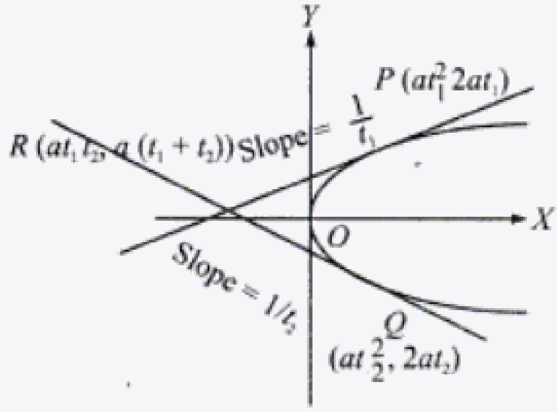
1. The *parametric equations* of the parabola or the coordinates of any point on it are $x = at^2$, $y = 2at$.
2. The *tangent* to the parabola at (x', y') is $yy' = 2a(x + x')$ and that at $(at^2, 2at)$ is $ty = x + at^2$.
3. The condition that the line $y = mx + c$ is a tangent to the parabola is $c = a/m$ and the equation of any tangent to it (not parallel to the y-axis) is therefore $y = mx + (a/m)$.
4. The *chord of contact* (defined as in circles) of (x', y') w.r.t. the parabola is $yy' = 2a(x + x')$.
5. The *polar* (defined as in circle) of (x', y') w.r.t. the parabola is $yy' = 2a(x + x')$.
6. The *chord with mid-Point* (x', y') of the parabola is $T = S'$, where $T = yy' - 2a(x + x')$ and $S' = y'^2 - 4ax'$.
7. The equation of the *pair of tangents* from (x', y') to the parabola is $T^2 = SS'$. Where $S = y^2 - 4ax$.
8. The *normal* at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^3$. If m is the slope of this normal, then its equation is $y = mx - 2am - am^3$, which is the normal to the parabola at $(am^2, -2am)$.
9. A *diameter* of the parabola is the locus of the middle points of a system of parallel chords of the parabola and the equation of a diameter is $y = 2a/m$ where m is the slope of the parallel chords which are bisected by it.
10. The equation of a chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.
11. If the chord joining the points having parameters t_1 and t_2 passes through the focus, then $t_1 t_2 = -1$.
12. If the coordinates of one end of a focal chord are $(at^2, 2at)$, then the coordinates of the other end are $(a/t^2, -2a/t)$.
13. For the end of the latus rectum, the values of the parameters t are ± 1.
14. The tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2; 2at_2)$ intersect at $(at_1 t_2, a(t_1 + t_2))$.
15. The tangents at the extremities of any focal chord intersect at right angles on the directrix.
16. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.
17. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.
18. The circle described on any focal chord of a parabola as diameter touches the directrix.

OPTICAL PROPERTY OF PARABOLA

(a) A ray parallel to the axis of the parabola after reflection from its internal surface passes through the focus.

(b) If a point is at a minimum distance from a parabola, then this point must lie on a normal to the parabola through this point.

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is



$R \equiv (at_1t_2,\ a(t_1 + t_2))$.

$\left(\text{i.e.} \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.

The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is $(x + \alpha)^2 \tan^2 \alpha = y^2 - 4ax$

The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R. Then the area of triangle PQR is $\frac{1}{2}a^2(t_1 - t_2)^3$.

If the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, then $ln = am^2$.

If the line $\frac{x}{l} + \frac{y}{m} = 1$ touches the parabola $y^2 = 4a(x + b)$ then $m^2(l + b) + al^2 = 0$.

If the two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at point P, whose abscissa is not zero, then the tangent to each curve at P, make complementary angle with the x-axis.

If the line $x \cos \alpha + y \sin a = p$ touches the parabola $y^2 = 4ax$, then $p \cos \alpha + a \sin^2 \alpha = 0$ and the point of contact is $(a \tan^2 \alpha, -2a \tan \alpha)$

Tangents at the extremities of any focul chord of a parabola meet at right angle on the directrix.

Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

If the tangents at the points P and Q on a parabola meet in T, then ST is the geometric mean between SP and SQ, i.e., $ST^2 = SP \cdot SQ$.

POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =$ or < 0, respectively.

NUMBER OF TANGENTS DRAWN FROM A POINT TO A PARABOLA

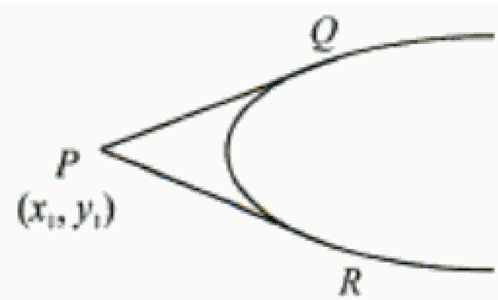
Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$,

where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1^2 - 4ax_1$

and $T \equiv yy_1 - 2a(x + x_1)$



EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. Point Form The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

2. Parametric Form The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3.$$

3. Slope Form The equation of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx - 2am - am^3.$$

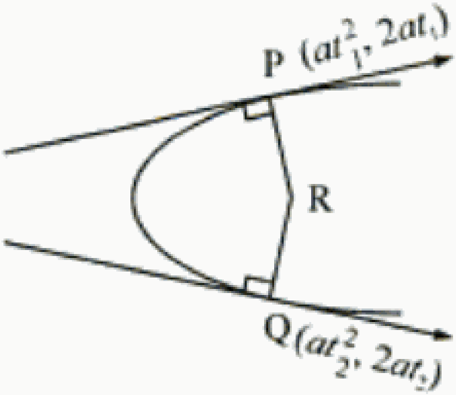
Note: The coordinates of the point of contact are $(am^2, -2am)$.

Condition for Normality The line $y = mx + c$ is a normal to the parabola

$$y^2 = 4ax \text{ if } c = -2am - am^3.$$

POINT OF INTERSECTION OF NORMALS

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

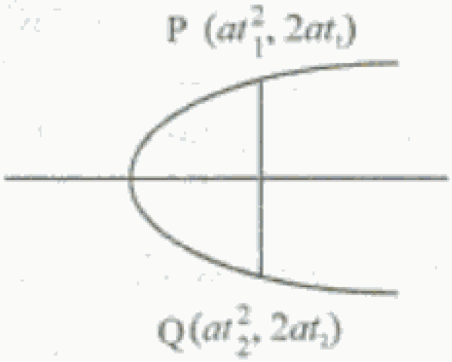


$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)].$$

If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1},$$

Note that PQ is normal to the parabola at P and not at Q.



If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$.

CO-NORMAL POINTS

Any three points on a parabola normals at which pass through a common point are called co-normal points

If three normals are drawn through a point (h, k), then their slopes are the roots of the cubic :

$$k = mh - 2am - am^3$$

(i) The sum of the slopes of the normals at co-normal points is zero, i.e. $m_1 + m_2 + m_3 = 0$.

(ii) The sum of the ordinates of the co-normal points is zero (i.e. $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$).

(iii) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola [the vertices of the triangle formed by the co-normal points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. Thus, y-coordinate of the centroid becomes

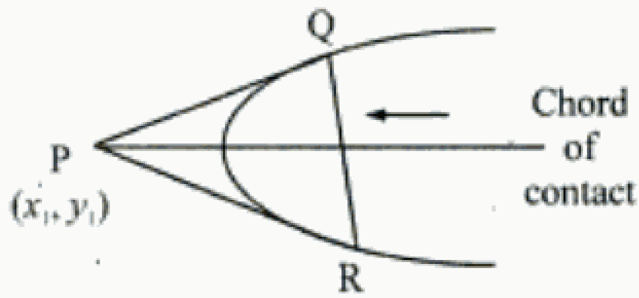
$$\frac{-2a(m_1 + m_2 + m_3)}{3} = \frac{-2a}{3} \times 0 = 0.$$

Hence, the centroid lies on the x-axis, i.e. axis of the parabola.]

(iv) If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$.

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.

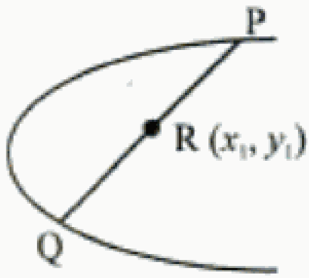


CHORD WITH A GIVEN MID POINT

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1$$

where $T \equiv yy_1 - 2a(x + x_1)$ and $S_1 \equiv y_1^2 - 4ax$.



at L, x = a $\therefore \frac{a\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$\Rightarrow y = \frac{b}{\sin\theta}(1 - \cos\theta)$

$\Rightarrow AL = \frac{b}{\sin\theta}(1 - \cos\theta)$

at , x = - a $\Rightarrow y = \frac{b}{\sin\theta}(1 + \cos\theta)$

$\Rightarrow A'M = \frac{b}{\sin\theta}(1 + \cos\theta)$

thus $AL.\ A'M = \frac{b^2}{\sin^2\theta}(1 - \cos^2\theta) = b^2$.

Question

If a, b, c are in A.P., a, x, b are in G.P. and b, y, c are in G.P., the point (x, y) lies on

(a) a straight line (b) a circle

(c) an ellipse (d) a hyperbola

Ans. (b)

Solution We have $2b = a + c$, $x^2 = ab$, $y^2 = bc$ so that $x^2 + y^2 = b(a + c) = 2b^2$ which is a circle.

Question

The second degree equation $x^2 + 3xy + 2y^2 + 3x + 5y + 2 = 0$ represents

a) parabola
b) ellipse
c) hyperbola
d) pair of straight lines

Solution

Ans (d)

Here a=1, $h = \frac{3}{2}$, b=2, $g = \frac{3}{2}$, $f = \frac{5}{2}$, c=2

thus $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 1.(2)(2) + 2\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)$$

$$-1\left(\frac{5}{2}\right)^2 - 2\left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)^2 = 0$$

thus the second dgree equation represents pair of straight lines.

Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$

Answer

The given equation is $\frac{x^2}{16}-\frac{y^2}{9}=1$ or $\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$.

On comparing this equation with the standard equation of hyperbola i.e., $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we obtain $a = 4$ and $b = 3$.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = 4^2 + 3^2 = 25$

$\Rightarrow c = 5$

Therefore,

The coordinates of the foci are (±5, 0).

The coordinates of the vertices are (±4, 0).

Eccentricity, $e=\frac{c}{a}=\frac{5}{4}$

Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times 9}{4}=\frac{9}{2}$

Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $\frac{y^2}{9}-\frac{x^2}{27}=1$

Answer

The given equation is $\frac{y^2}{9}-\frac{x^2}{27}=1$ or $\frac{y^2}{3^2}-\frac{x^2}{\left(\sqrt{27}\right)^2}=1$.

On comparing this equation with the standard equation of hyperbola i.e., $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we obtain a = 3 and $b=\sqrt{27}$.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = 3^2+\left(\sqrt{27}\right)^2 = 9+27=36$

$\Rightarrow c=6$

Therefore,

The coordinates of the foci are (0, ±6).

The coordinates of the vertices are (0, ±3).

Eccentricity, $e=\frac{c}{a}=\frac{6}{3}=2$

Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times 27}{3}=18$

Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $9y^2 - 4x^2 = 36$

Answer

The given equation is $9y^2 - 4x^2 = 36$.

It can be written as

$9y^2 - 4x^2 = 36$

Or, $\frac{y^2}{4} - \frac{x^2}{9} = 1$

Or, $\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$ ...(1)

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we obtain $a = 2$ and $b = 3$.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = 4 + 9 = 13$

$\Rightarrow c = \sqrt{13}$

Therefore,

The coordinates of the foci are $\left(0,\ \pm\sqrt{13}\right)$.

The coordinates of the vertices are $(0,\ \pm 2)$

Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{2} = 9$

Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 576$

Answer

The given equation is $16x^2 - 9y^2 = 576$.

It can be written as

$16x^2 - 9y^2 = 576$

$$\Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1$$

$$\Rightarrow \frac{x^2}{6^2} - \frac{y^2}{8^2} = 1 \qquad ...(1)$$

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we obtain $a = 6$ and $b = 8$.

We know that $a^2 + b^2 = c^2$.

$$\therefore c^2 = 36 + 64 = 100$$

$$\Rightarrow c = 10$$

Therefore,

The coordinates of the foci are $(\pm 10, 0)$.

The coordinates of the vertices are $(\pm 6, 0)$.

Eccentricity, $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 64}{6} = \frac{64}{3}$

Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $5y^2 - 9x^2 = 36$

Answer

The given equation is $5y^2 - 9x^2 = 36$.

$$\Rightarrow \frac{y^2}{\left(\frac{36}{5}\right)} - \frac{x^2}{4} = 1$$

$$\Rightarrow \frac{y^2}{\left(\frac{6}{\sqrt{5}}\right)^2} - \frac{x^2}{2^2} = 1 \qquad ...(1)$$

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we obtain $a = \frac{6}{\sqrt{5}}$ and $b = 2$.

We know that $a^2 + b^2 = c^2$.

$$\therefore c^2 = \frac{36}{5} + 4 = \frac{56}{5}$$

$$\Rightarrow c = \sqrt{\frac{56}{5}} = \frac{2\sqrt{14}}{\sqrt{5}}$$

Therefore, the coordinates of the foci are $\left(0,\ \pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$

The coordinates of the vertices are $\left(0,\ \pm\frac{6}{\sqrt{5}}\right)$.

$$\text{Eccentricity, } e=\frac{c}{a}=\frac{\left(\frac{2\sqrt{14}}{\sqrt{5}}\right)}{\left(\frac{6}{\sqrt{5}}\right)}=\frac{\sqrt{14}}{3}$$

Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times 4}{\left(\frac{6}{\sqrt{5}}\right)}=\frac{4\sqrt{5}}{3}$

Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $49y^2 - 16x^2 = 784$

Answer

The given equation is $49y^2 - 16x^2 = 784$.

It can be written as

$49y^2 - 16x^2 = 784$

$$\text{Or, } \frac{y^2}{16} - \frac{x^2}{49} = 1$$

$$\text{Or, } \frac{y^2}{4^2} - \frac{x^2}{7^2} = 1 \qquad \ldots(1)$$

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we obtain $a = 4$ and $b = 7$.

We know that $a^2 + b^2 = c^2$.

$$\therefore c^2 = 16 + 49 = 65$$

$$\Rightarrow c = \sqrt{65}$$

Therefore,

The coordinates of the foci are $\left(0, \pm\sqrt{65}\right)$.

The coordinates of the vertices are $(0, \pm 4)$.

$$\text{Eccentricity, } e = \frac{c}{a} = \frac{\sqrt{65}}{4}$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{2 \times 49}{4} = \frac{49}{2}$$

Question

Find the equation of the hyperbola satisfying the give conditions: Vertices (±2, 0), foci (±3, 0)

Answer

Vertices (±2, 0), foci (±3, 0)

Here, the vertices are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the vertices are (±2, 0), $a = 2$.

Since the foci are (±3, 0), $c = 3$.

We know that $a^2 + b^2 = c^2$.

$\therefore 2^2 + b^2 = 3^2$

$b^2 = 9 - 4 = 5$

Thus, the equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

Question

Find the equation of the hyperbola satisfying the give conditions: Vertices (0, ±5), foci (0, ±8)

Answer

Vertices (0, ±5), foci (0, ±8)

Here, the vertices are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the vertices are (0, ±5), $a = 5$.

Since the foci are (0, ±8), $c = 8$.

We know that $a^2 + b^2 = c^2$.

$\therefore 5^2 + b^2 = 8^2$

$b^2 = 64 - 25 = 39$

Thus, the equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{39} = 1$.

Question

Find the equation of the hyperbola satisfying the give conditions: Vertices (0, ±3), foci (0, ±5)

Answer

Vertices (0, ±3), foci (0, ±5)

Here, the vertices are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the vertices are (0, ±3), $a = 3$.

Since the foci are (0, ±5), $c = 5$.

We know that $a^2 + b^2 = c^2$.

$\therefore 3^2 + b^2 = 5^2$

$\Rightarrow b^2 = 25 - 9 = 16$

Thus, the equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

Question

Find the equation of the hyperbola satisfying the give conditions: Foci (±5, 0), the transverse axis is of length 8.

Answer

Foci (±5, 0), the transverse axis is of length 8.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the foci are (±5, 0), $c = 5$.

Since the length of the transverse axis is 8, $2a = 8 \Rightarrow a = 4$.

We know that $a^2 + b^2 = c^2$.

$\therefore 4^2 + b^2 = 5^2$

$\Rightarrow b^2 = 25 - 16 = 9$

Thus, the equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Question

Find the equation of the hyperbola satisfying the give conditions: Foci (0, ±13), the conjugate axis is of length 24.

Answer

Foci (0, ±13), the conjugate axis is of length 24.

Here, the foci are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the foci are (0, ±13), $c = 13$.

Since the length of the conjugate axis is 24, $2b = 24 \Rightarrow b = 12$.

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 12^2 = 13^2$

$\Rightarrow a^2 = 169 - 144 = 25$

Thus, the equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{144} = 1$.

Question

Find the equation of the hyperbola satisfying the give conditions: Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.

Answer

Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the foci are $(\pm 3\sqrt{5}, 0)$, $c = \pm 3\sqrt{5}$.

Length of latus rectum = 8

$\Rightarrow \frac{2b^2}{a} = 8$

$\Rightarrow b^2 = 4a$

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 4a = 45$

$\Rightarrow a^2 + 4a - 45 = 0$

$\Rightarrow a^2 + 9a - 5a - 45 = 0$

$\Rightarrow (a + 9)(a - 5) = 0$

$\Rightarrow a = -9, 5$

Since a is non-negative, $a = 5$.

$\therefore b^2 = 4a = 4 \times 5 = 20$

Thus, the equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{20} = 1$

Question

Find the equation of the hyperbola satisfying the give conditions: Foci (±4, 0), the latus rectum is of length 12

Answer

Foci (±4, 0), the latus rectum is of length 12.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the foci are (±4, 0), $c = 4$.

Length of latus rectum = 12

$\Rightarrow \frac{2b^2}{a} = 12$

$\Rightarrow b^2 = 6a$

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 6a = 16$

$\Rightarrow a^2 + 6a - 16 = 0$

$\Rightarrow a^2 + 8a - 2a - 16 = 0$

$\Rightarrow (a + 8)(a - 2) = 0$

$\Rightarrow a = -8, 2$

Since a is non-negative, $a = 2$.

$\therefore b^2 = 6a = 6 \times 2 = 12$

Thus, the equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{12}=1$

Question

Find the equation of the hyperbola satisfying the give conditions: Vertices (±7, 0), $e=\frac{4}{3}$

Answer

Vertices (±7, 0), $e=\frac{4}{3}$

Here, the vertices are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Since the vertices are (±7, 0), a = 7.

It is given that $e=\frac{4}{3}$

$$\therefore \frac{c}{a}=\frac{4}{3} \qquad \left[e=\frac{c}{a}\right]$$

$$\Rightarrow \frac{c}{7}=\frac{4}{3}$$

$$\Rightarrow c=\frac{28}{3}$$

We know that $a^2 + b^2 = c^2$.

$$\therefore 7^2+b^2=\left(\frac{28}{3}\right)^2$$

$$\Rightarrow b^2=\frac{784}{9}-49$$

$$\Rightarrow b^2=\frac{784-441}{9}=\frac{343}{9}$$

Thus, the equation of the hyperbola is $\frac{x^2}{49}-\frac{9y^2}{343}=1$

Question

Find the equation of the hyperbola satisfying the give conditions: Foci $\left(0,\ \pm\sqrt{10}\right)$, passing through (2, 3)

Answer

Foci $\left(0,\ \pm\sqrt{10}\right)$, passing through (2, 3)

Here, the foci are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

Since the foci are $\left(0,\ \pm\sqrt{10}\right)$, $c = \sqrt{10}$.

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + b^2 = 10$

$\Rightarrow b^2 = 10 - a^2$ ... (1)

Since the hyperbola passes through point (2, 3),

$$\frac{9}{a^2}-\frac{4}{b^2}=1 \qquad ...(2)$$

From equations (1) and (2), we obtain

$$\frac{9}{a^2}-\frac{4}{\left(10-a^2\right)}=1$$

$$\Rightarrow 9\left(10-a^2\right)-4a^2=a^2\left(10-a^2\right)$$

$$\Rightarrow 90-9a^2-4a^2=10a^2-a^4$$
$$\Rightarrow a^4-23a^2+90=0$$
$$\Rightarrow a^4-18a^2-5a^2+90=0$$
$$\Rightarrow a^2\left(a^2-18\right)-5\left(a^2-18\right)=0$$
$$\Rightarrow \left(a^2-18\right)\left(a^2-5\right)=0$$
$$\Rightarrow a^2=18 \text{ or } 5$$

In hyperbola, $c > a$, i.e., $c^2 > a^2$

$\therefore a^2 = 5$

$\Rightarrow b^2 = 10 - a^2 = 10 - 5 = 5$

Thus, the equation of the hyperbola is $\frac{y^2}{5}-\frac{x^2}{5}=1$

Question

Let $S(-1,1)$ be the focus and $P(x,y)$ be a point on the hyperbola Draw PM perpendicular from P on the directrix. Then, by definition.

$$SP = ePM$$
$$\Rightarrow \quad SP^2 = e^2PM^2$$
$$\Rightarrow \quad (x+1)^2+(y-1)^2=(3)^2\left[\frac{x-y+3}{\sqrt{1^2+(-1)^2}}\right]^2 \qquad [\because e=3]$$
$$\Rightarrow \quad x^2+1+2x+y^2+1-2y=\frac{9[x-y+3]^2}{2}$$
$$\Rightarrow \quad 2\left[x^2+y^2+2x-2y+2\right]=9[x-y+3]^2$$
$$\Rightarrow \quad 2x^2+2y^2+4x-4y+4=9\left[x^2(-y)^2+3^2+2\times x\times(-y)+2\times(-y)\times 3+2\times 3\times x\right]$$
$$\Rightarrow \quad 2x^2+2y^2+4x-4y-4=9\left[x^2+y^2+9-2xy-6y+6x\right]$$
$$\Rightarrow \quad 2x^2+2y^2+4x-4y+4=9x^2+9y^2+81-18xy-54y+4y+81-4=0$$
$$\Rightarrow \quad 7x^2+7y^2-18xy+50x-50y+77=0$$

This is the required equation of the hyperbota

Question

Let $S(0,3)$ be the focus and $P(x,y)$ be a point on the hyperbola.
Draw PM perpendicular from P on the directrix. Then, by definition

$$sP = ePM$$
$$\Rightarrow \quad sP^2 = e^2PM^2$$
$$\Rightarrow \quad (x-0)^2 + (y-3)^2 = 2^2\left[\frac{x+y-1}{\sqrt{1^2+1^2}}\right]^2 \qquad [\because e = 2]$$
$$\Rightarrow \quad x^2 + y^2 + 9 - 6y = \frac{4[x+y-1]^2}{2}$$
$$\Rightarrow \quad x^2 + y^2 - 6y + 9 = 2(x+y-1)^2$$
$$\Rightarrow \quad x^2 + y^2 - 6y + 9 = 2\left[x^2 + y^2 + (-1)^2 + 2xy + 2 \times y \times (-1) + 2 \times (-1) \times x\right]$$
$$\Rightarrow \quad x^2 + y^2 - 6y + 9 = 2\left[x^2 + y^2 + 1 + 2xy - 2y - 2x\right]$$
$$\Rightarrow \quad x^2 + y^2 - 6y + 9 = 2x^2 + 2y^2 + 2 + 4xy - 4y - 4x$$
$$\Rightarrow \quad 2x^2 - x^2 + 2y^2 - y^2 + 4xy - 4x - 4y + 6y + 2 - 9 = 0$$
$$\Rightarrow \quad x^2 + y^2 + 4xy - 4x + 2y - 7 = 0$$

This is the required equation of the hyperbola.

Let $S(1,1)$ be the focus and $P(x,y)$ be a point on the hyperbola.
Draw PM perpendicular from P on the directrix. Then, by definition

$$sP = ePM$$
$$\Rightarrow \quad sP^2 = e^2PM^2$$
$$\Rightarrow \quad (x-1)^2 + (y-1)^2 = 2^2\left[\frac{3x+4y+8}{\sqrt{3^2+4^2}}\right]^2 \qquad [\because e = 2]$$
$$\Rightarrow \quad x^2 + 1 - 2x + y^2 + 1 - 2y = 4\left[\frac{3x+4y+8}{\sqrt{25}}\right]$$
$$\Rightarrow \quad x^2 + y^2 - 2x - 2y + 2 = \frac{4(3x+4y+8)^2}{25}$$
$$\Rightarrow \quad 25x^2 + 25y^2 - 50x - 50y + 50 = 4(3x+4y+8)^2$$
$$\Rightarrow \quad 25x^2 + 25y^2 - 50x - 50y + 50 = 4\left[9x^2 + 16y^2 + 6y + 24xy + 64y + 48x\right]$$
$$\Rightarrow \quad 25x^2 + 25y^2 - 50x - 50y + 50 = 36x^2 + 64y^2 + 256 + 96xy + 256y + 192x$$
$$\Rightarrow \quad 36x^2 - 25x^2 + 64y^2 - 25y^2 + 96xy + 192x + 50x + 256y + 50y + 256 - 50 = 0$$
$$\Rightarrow \quad 11x^2 + 39y^2 + 96xy + 242x + 306y + 206 = 0$$

This is the required equation of the hyperbola.

Question

Let $S(1,1)$ be the focus and $P(x,y)$ be a point on the hyperbola.
Draw PM perpendicular from P on the directrix. Then, by definition

$$sP = ePM$$
$$\Rightarrow \quad sP^2 = e^2PM^2$$
$$\Rightarrow \quad (x-1)^2 + (y-1)^2 = \left(\sqrt{3}\right)^2 \left[\frac{2x+y-1}{\sqrt{2^2+1^2}}\right]^2 \qquad [\because e = 2]$$
$$\Rightarrow \quad x^2 + 1 - 2x + y^2 + 1 - 2y = \frac{3[2x+y-1]^2}{5}$$
$$\Rightarrow \quad 5\left[x^2 + y^2 - 2x - 2y + 2\right] = 3(2x+y-1)^2$$
$$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 10y + 10 = 3\left[(2x)^2 + y^2 + (-1)^2 + 2\times 2x \times y + 2\times y \times (-1) + 2\times(-1)\times 2x\right]$$
$$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 10y + 10 = 3\left[4x^2 + y^2 + 1 + 4xy - 2y - 4x\right]$$
$$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 10y + 10 = 12x^2 + 3y^2 + 3 + 12xy - 6y - 12x$$
$$\Rightarrow \quad 12x^2 - 5x^2 + 3y^2 - 5y^2 + 12xy - 12x + 10x - 6y + 10y + 3 - 10 = 0$$
$$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 4y - 7 = 0$$

This is the required equation of the hyperbola.

Question

Let $S(2,-1)$ be the focus and $P(x,y)$ be a point on the hyperbola.
Draw PM perpendicular from P on the directrix. Then, by definition

$$sP = ePM$$
$$\Rightarrow \quad sP^2 = e^2PM^2$$
$$\Rightarrow \quad (x-2)^2 + (y+1)^2 = 2^2\left[\frac{2x+3y-1}{\sqrt{2^2+3^2}}\right]^2 \qquad [\because e = 2]$$
$$\Rightarrow \quad x^2 + 4 - 4x + y^2 + 1 + 2y = \frac{4[2x+3y-1]^2}{13}$$
$$\Rightarrow \quad 13\left[x^2 + y^2 - 4x + 2y + 5\right] = 4(2x+3y-1)^2$$
$$\Rightarrow \quad 13x^2 + 13y^2 - 52x + 26y + 65 = 4[2x+3y-1]^2$$
$$\Rightarrow \quad 13x^2 + 13y^2 - 52x + 26y + 65 = 4\left[(2x)^2 + (3y)^2 + (-1)^2 + 2\times 2x\times 3y + 2\times 3y\times(-1) + 2\times(-1)\times 2x\right]$$
$$\Rightarrow \quad 13x^2 + 13y^2 - 52x + 26y + 65 = 4\left[4x^2 + 9y^2 + 1 + 12xy - 6y - 4x\right]$$
$$\Rightarrow \quad 13x^2 + 13y^2 - 52x + 26y + 65 = 16x^2 + 36y^2 + 4 + 48xy - 24y - 16x$$
$$\Rightarrow \quad 16x^2 - 13x^2 + 36y^2 - 13y^2 + 48xy - 16x + 52x - 24y - 26y + 4 - 65 = 0$$
$$\Rightarrow \quad 3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0$$

This is the required equation of the hyperbola.

Question

Let $S(a,0)$ be the focus and $P(x,y)$ be a point on the hyperbola.
Draw PM perpendicular from P on the directrix. Then, by definition

$$sP = ePM$$
$$\Rightarrow \quad sP^2 = e^2PM^2$$
$$\Rightarrow \quad (x-a)^2 + (y-0)^2 = \left(\frac{4}{3}\right)^2\left[\frac{2x-y+a}{\sqrt{2^2+(-1)^2}}\right]^2 \qquad \left[\because e = \frac{4}{3}\right]$$
$$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 = \frac{16}{9}\times\frac{[2x-y+a]^2}{5}$$
$$\Rightarrow \quad 45\left[x^2+y^2-2ax+a^2\right] = 16[2x-y+a]^2$$
$$\Rightarrow \quad 45x^2+45y^2-90ax+45a^2 = 16\left[(2x)^2+(-y)^2+a^2+2\times2x(-y)+2\times(-y)\times a+2\times a\times 2x\right]$$
$$\Rightarrow \quad 45x^2+45y^2-90ax+45a^2 = 16\left[4x^2+y^2+a^2-4xy-2ay+4ax\right]$$
$$\Rightarrow \quad 45x^2+45y^2-90ax+45a^2 = 64x^2+16y^2+16a^2-64xy-32ay+64ax$$
$$\Rightarrow \quad 64x^2-45x^2+16y^2-45y^2-64xy+64ax+90ax-32ay+16a^2-45a^2 = 0$$
$$\Rightarrow \quad 19x^2-29y^2-64xy+154ax-32ay-29a^2 = 0$$

This is the required equation of the hyperbola.

Question

Let $S(2,2)$ be the focus and $P(x,y)$ be a point on the hyperbola.
Draw PM perpendicular from P on the directrix. Then, by definition

$$sP = ePM$$
$$\Rightarrow \quad sP^2 = e^2PM^2$$
$$\Rightarrow \quad (x-2)^2 + (y-2)^2 = 2^2\left[\frac{x+y-9}{\sqrt{1^2+1^2}}\right]^2 \qquad \left[\because e = \frac{4}{3}\right]$$
$$\Rightarrow \quad x^2+4-4x+y^2+4-4y = \frac{4[x+y-9]^2}{2}$$
$$\Rightarrow \quad x^2+y^2-4x-4y+8 = 2[x+y-9]^2$$
$$\Rightarrow \quad x^2+y^2-4x-4y+8 = 2\left[x^2+y^2+(-9)^2+2\times x\times y+2\times y\times(-9)+2\times(-9)\times x\right]$$
$$\Rightarrow \quad x^2+y^2-4x-4y+8 = 2\left[x^2+y^2+81+2xy-18y+18x\right]$$
$$\Rightarrow \quad x^2+y^2-4x-4y+8 = \left[2x^2+2y^2+162+4xy-36y-36x\right]$$
$$\Rightarrow \quad 2x^2-x^2+2y^2-y^2+4xy-36x+4x-36y+4y+162-8 = 0$$
$$\Rightarrow \quad x^2+y^2+4xy-32x-32y+154 = 0$$

This is the required equation of the hyperbola.

Question

We have,

$$9x^2 - 16y^2 = 144$$

$$\Rightarrow \quad \frac{9x^2}{144} - \frac{16y^2}{144} = 1$$

$$\Rightarrow \quad \frac{x^2}{16} - \frac{y^2}{9} = 1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a^2 = 16$ and $b^2 = 9$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{9}{16}}$$

$$= \sqrt{\frac{25}{16}}$$

$$= \frac{5}{4}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$ i.e., $(\pm 5, 0)$

Equations of the directrices: The equations of the directrices are

$$x = \pm\frac{a}{e} \text{ i.e., } x = \pm\frac{16}{5}$$

$$\therefore \quad 5x = \pm 16$$

$$\Rightarrow \quad 5x \mp 16 = 0$$

Length of latus-rectum: The length of the latus-rectum

$$= \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$$

We have,

$$16x^2 - 9y^2 = -144$$

$$\Rightarrow \quad \frac{16x^2}{144} - \frac{9y^2}{144} = -1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{16} = -1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, where $a^2 = 9$ and $b^2 = 16$

$\therefore \quad a = 3$ and $b = 4$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 + \frac{a^2}{b^2}}$$

$$= \sqrt{1 + \frac{9}{16}}$$

$$= \sqrt{\frac{25}{16}}$$

$$= \frac{5}{4}$$

Foci: The coordinates of the foci are $(0, \pm be)$.

$$\therefore \quad (0, \pm be) = \left(0, \pm 4 \times \frac{5}{4}\right)$$

$$= (0, \pm 5)$$

$\therefore$ the coordinates of the foci are $(0, \pm 5)$

Equations of the directrices: The equations of the directrices are

$$y = \frac{\pm b}{e}$$

$$\Rightarrow \quad y = \pm \frac{4}{\frac{5}{4}} = \pm \frac{16}{5}$$

$$\Rightarrow \quad 5y \mp 16 = 0$$

Latus-rectum: The length of the latus-rectum

$$= \frac{2a^2}{b}$$

$$= \frac{2 \times 9}{4} = \frac{9}{2}$$

Question

We have,

$$4x^2 - 3y^2 = 36$$

$$\Rightarrow \quad \frac{4x^2}{36} - \frac{3y^2}{36} = 1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{12} = 1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, where $a^2 = 9$ and $b^2 = 12$

$\therefore \quad a = 3$ and $b = \sqrt{12} = 2\sqrt{3}$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{12}{9}}$$

$$= \sqrt{1 + \frac{4}{3}}$$

$$= \sqrt{\frac{7}{3}}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$.

$$\therefore \quad \pm ae = \pm 3 \times \sqrt{\frac{7}{3}}$$
$$= \pm 3 \times \frac{\sqrt{7}}{\sqrt{3}}$$
$$= \pm\sqrt{3} \times \sqrt{7}$$
$$= \pm\sqrt{21}$$
$$\therefore \quad (\pm ae, 0) = (\pm\sqrt{21}, 0)$$

$\therefore$ the coordinates of the foci are $(\pm\sqrt{21}, 0)$

Equations of the directrices: The equations of the directrices are

$$x = \frac{\pm a}{e}$$
$$\therefore \quad x = \pm 3 \times \frac{1}{\frac{\sqrt{7}}{\sqrt{3}}}$$
$$= \pm \frac{3\sqrt{3}}{\sqrt{7}}$$
$$\Rightarrow \quad \sqrt{7}x \mp 3\sqrt{3} = 0$$

$\therefore$ The equations of the directrices are $\sqrt{7}x \mp 3\sqrt{3} = 0$

Latus-rectum: The length of the latus-rectum

$$= \frac{2b^2}{a} = \frac{2 \times 12}{3} = 8$$

We have,

$$3x^2 - y^2 = 4$$

$$\Rightarrow \quad \frac{3x^2}{4} - \frac{y^2}{4} = 1$$

$$\Rightarrow \quad \frac{x^2}{\frac{4}{3}} - \frac{y^2}{4} = 1$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{2}{\sqrt{3}}\right)^2} - \frac{y^2}{2^2} = 1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a = \frac{2}{\sqrt{3}}$ and $b = 2$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1+\frac{b^2}{a^2}}$$
$$= \sqrt{1+\frac{4}{\frac{4}{3}}}$$
$$= \sqrt{1+3}$$
$$= \sqrt{4}$$
$$= 2$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$\therefore \quad \pm ae = \pm\frac{2}{\sqrt{3}}\times 2 = \pm\frac{4}{\sqrt{3}}$

The coordinates of the foci are $\left(\pm\frac{4}{\sqrt{3}}, 0\right)$

Equations of the directirices: The equations of the directrices are

$$x = \pm\frac{a}{e}$$
$$= \pm\frac{\frac{2}{\sqrt{3}}}{2}$$
$$= \pm\frac{1}{\sqrt{3}}$$

$\Rightarrow \quad \sqrt{3}x \mp 1 = 0$

Latus-rectum: The length of the latus-rectum $= \frac{2b^2}{a}$

$\therefore \quad \frac{2b^2}{a} = 2\times\frac{4}{\frac{2}{\sqrt{3}}}$

$$= 4\sqrt{3}$$

Question

We have,

$$25x^2 - 36y^2 = 225$$

$$\Rightarrow \quad \frac{25x^2}{225} - \frac{36^2}{225} = 1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{4y^2}{25} = 1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{\frac{25}{4}} = 1$$

$$\Rightarrow \quad \frac{x^2}{(3)^2} - \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a = 3$ and $b = \frac{5}{2}$

Length of the transverse axis: The length of the transverse axis

$= 2a$

$= 2 \times 3 = 6$

Length of the conjugate axis: The length of the conjugate axis is

$$2b = 2 \times \frac{5}{2} = 5$$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{\frac{25}{4}}{9}}$$

$$= \sqrt{1 + \frac{25}{36}}$$

$$= \sqrt{\frac{61}{36}}$$

$$= \frac{\sqrt{61}}{6}$$

Length of LR$= \frac{2b^2}{a} = \frac{25}{6}$

Foci $(\pm\frac{\sqrt{61}}{2}, 0)$

Question

We have,

$$16x^2 - 9y^2 + 32x + 36y - 164 = 0$$

$$\Rightarrow \quad 16x^2 + 32x - 9y^2 + 36y - 14 = 0$$

$$\Rightarrow \quad 16\left(x^2 + 2x\right) - 9\left(y^2 + 4y\right) - 164 = 0$$

$$\Rightarrow \quad 16\left[x^2 + 2x + 1 - 1\right] - 9\left[y^2 - 4y + 4 - 4\right] - 164 = 0$$

$$\Rightarrow \quad 16\left[(x+1)^2 - 1\right] - 9\left[(y-2)^2 - 4\right] - 164 = 0$$

$$\Rightarrow \quad 16(x+1)^2 - 16 - 9(y-2)^2 + 36 - 164 = 0$$

$$\Rightarrow \quad 16(x+1)^2 - 9(y-2)^2 + 20 - 164 = 0$$

$$\Rightarrow \quad 16(x+1)^2 - 9(y-2)^2 - 144 = 0$$

$$\Rightarrow \quad 16(x+1)^2 - 9(y-2)^2 = 144$$

$$\Rightarrow \quad \frac{16(x+1)^2}{144} - \frac{9(y-2)^2}{144} = 1$$

$$\Rightarrow \quad \frac{(x+1)^2}{9} - \frac{(y-2)}{16} = 1 \qquad \text{---(i)}$$

Shifting the origin at $(-1,2)$ without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and y,

We have,

$$x = X - 1 \text{ and } y = Y + 2 \qquad \text{---(ii)}$$

This is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 16$. so, We have,

Centre: The coordinates of the centre w.r.t the new axes are $(X = 0, Y = 0)$

$\therefore \quad x = -1$ and $y = 2$ [Using equation (ii)]

So, the coordinates of the centre w.r.t the old axes are $(-1, 2)$.

Eccentricity: The ecentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$
$$= \sqrt{1 + \frac{16}{9}}$$
$$= \sqrt{\frac{25}{9}}$$
$$= \frac{5}{3}$$

Foci: The coordinates of the foci with respect to the new axes are given by $(X = \pm ae, Y = 0)$ i.e., $(X = \pm 5, Y = 0)$.

Putting $X = \pm 5$ and $Y = 0$ in equation (ii), we get

$x = \pm 5 - 1$ and $y = 0 + 2$

$\Rightarrow \quad x = 4, -6$ and $y = 2$

Equation of the directix: The equations of the directirx are

$$X = \pm\frac{a}{e}$$
$$= \pm\frac{3}{\frac{5}{3}}$$
$$X = \pm\frac{9}{5}$$

Putting $X = \pm\frac{9}{5}$ in equation (ii), we get

$$x = \pm\frac{9}{5} - 1$$
$$\Rightarrow \quad x = \frac{\pm 9 - 5}{5}$$
$$\Rightarrow \quad x = \frac{4}{5} \text{ and } x = \frac{-14}{5}$$
$$\Rightarrow \quad 5x - 4 = 0 \text{ and } 5x + 14 = 0$$

So, the equations of the directrices w.r.t the old axes are

$5x - 4 = 0$ and $5x + 14 = 0$.

We have,

$$x^2 - y^2 + 4x = 0$$

$$\Rightarrow \quad x^2 + 4x - y^2 = 0$$

$$\Rightarrow \quad x^2 + 4x + 4 - 4 - y^2 = 0$$

$$\Rightarrow \quad (x+2)^2 - y^2 = 4$$

$$\Rightarrow \quad \frac{(x+2)^2}{4} - \frac{y^2}{4} = 1 \qquad \text{---(i)}$$

Shifting the origin at $(-2, 0)$ without rotating the axes and denoting the new coordinates w.r.t these axes by X and y,

We have,

$$x = X - 2 \text{ and } y = Y \qquad \text{---(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{4} - \frac{Y^2}{4} = 1 \qquad \text{---(ii)}$$

This is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$, where $a^2 = 4$ and $b^2 = 4$. so,

We have,

Centre: The coordinates of the centre w.r.t the new axes are $(X = 0, Y = 0)$

Putting $X = 0$ and $Y = 0$ in equation (ii), we get

$x = -2$ and $y = 0$.

So, the coordinates of the centre w.r.t the old axes are $(-2, 0)$.

Question

The eccentricity of the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$ is

(a) 5/4 (b) 4/5

(c) 9/16 (d) 16/9

Solution

(a). The given hyperbola can be written in the form

$$\frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1.$$

Here $a^2 = 16$ and $b^2 = 9$.

$$\therefore \quad e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow \boldsymbol{e = 5/4.}$$

Question

The number of tangents to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ through (4, 1) is

(a) 1 (b) 2

(c) 0 (d) 3

Solution

(b). Since $\left.\frac{x^2}{4} - \frac{y^2}{3} - 1\right]_{(4,1)} = \frac{16}{4} - \frac{1}{3} - 1 > 0$

$\therefore$ the point (4, 1) lies outside the hyperbola, hence the number of tangents through (4, 1) is **two.**

Question

The equation of common tangents to the parabola $y^2 = 8x$ and hyperbola $3x^2 - y^2 = 3$, is

(a) $2x \pm y + 1 = 0$ (b) $2x \pm y - 1 = 0$

(c) $x \pm 2y + 1 = 0$ (d) $x \pm 2y - 1 = 0$

Solution

(a). The equation of tangent to $y^2 = 8x$ is $y = mx + \frac{2}{m}$

Also, the equation of tangent to $\frac{x^2}{1} - \frac{y^2}{3} = 1$

$\Rightarrow \quad y = mx \pm \sqrt{m^2 - 3}$

On comparing, we get

$m = \pm 2$ or tangent as $2x \pm y + 1 = 0$.

Question

Let P $(a \sec \theta, b \tan \theta)$ and Q $(a \sec \phi, b \tan \phi)$ where $\theta + \phi = \pi/2$, be two points on the hyperbola $x^2/a^2 - y^2/b^2 = 1$. If (h, k) is the point of intersection of normals at P and Q, then k is equal to

(a) $\frac{a^2 + b^2}{a}$ (b) $-\left[\frac{a^2 + b^2}{a}\right]$

(c) $\frac{a^2 + b^2}{b}$ (d) $-\left[\frac{a^2 + b^2}{b}\right]$

Ans. (d)

Solution Equation of the tangent at P $(a \sec \theta, b \tan \theta)$ is

$\frac{x}{a} \sec \theta - \frac{y}{a} \tan \theta = 1.$

Therefore equation of the normal at P is

$$y - b \tan \theta = -\frac{a}{b} \sin \theta (x - a \sec \theta)$$

$\Rightarrow \qquad ax + b \text{ cosec } \theta\, y = (a^2 + b^2) \sec \theta \qquad$ (i)

Similarly the equation of the normal at Q ($a \sec \phi$, $b \sec \phi$) is

$$ax + b \text{ cosec } \phi\, y = (a^2 + b^2) \sec \phi \qquad \text{(ii)}$$

Subtracting (ii) from (i) we get $y = \dfrac{a^2 + b^2}{b} \cdot \dfrac{\sec\theta - \sec\phi}{\text{cosec } \theta - \text{cosec } \phi}$

So that $\quad k = y = \dfrac{a^2 + b^2}{b} \dfrac{\sec\theta - \sec(\pi/2 - \theta)}{\text{cosec } \theta - \text{cosec } (\pi/2 - \theta)} \qquad [\because \theta + \phi = \pi/2]$

$$= \frac{a^2 + b^2}{b} \frac{\sec\theta - \text{cosec } \theta}{\text{cosec } \theta - \sec\theta} = -\left[\frac{a^2 + b^2}{b}\right]$$

Question

A tangent to the hyperbola $y = \dfrac{x + 9}{x + 5}$ passing through the origin is

(a) $x + 25y = 0$ (b) $5x + y = 0$

(c) $5x - y = 0$ (d) $x - 25y = 0$

Solution

(c). $y = \dfrac{x+9}{x+5} = 1 + \dfrac{4}{x+5}$; $\dfrac{dy}{dx}$ at $(x_1, y_1) = -\dfrac{4}{(x_1+5)^2}$

Equation of tangent is $y - y_1 = -\dfrac{4}{(x_1+5)^2}(x - x_1)$

$$y - 1 - \frac{4}{x_1+5} = -\frac{4}{(x_1+5)^2}(x - x_1)$$

Since it passes through origin (0, 0)

$$-1 - \frac{4}{x_1+5} = \frac{4x_1}{(x_1+5)^2}$$

$\Rightarrow$ $(x_1 + 5)^2 + 4(x_1 + 5) + 4x_1 = 0$

$\Rightarrow$ $x_1^2 + 18x_1 + 45 = 0$

$\Rightarrow$ $(x_1 + 15)(x_1 + 3) = 0$

$\Rightarrow$ $x_1 = -15$ or $x_1 = -3$

So equation of tangent is

$$y - 1 - \frac{4}{(-15+5)} = -\frac{4}{(-15+5)^2}(x + 15)$$

$\Rightarrow$ $y - 1 + \dfrac{2}{5} = -\dfrac{1}{25}(x + 15)$

$\Rightarrow$ $y - \dfrac{3}{5} = -\dfrac{x}{25} - \dfrac{3}{5}$

$\Rightarrow$ $x + 25y = 0$

or $\quad y - 1 - \dfrac{4}{(-3+5)} = -\dfrac{4}{(-3+5)^2}(x + 3)$

$\Rightarrow$ $y - 1 - 2 = -(x + 3)$ or $x + y = 0$

Question

The point of intersection of two tangents to the hyperbola $x^2/a^2 - y^2/b^2 = 1$, the product of whose slopes is c^2, lies on the curve.

(a) $y^2 - b^2 = c^2(x^2 + a^2)$ (b) $y^2 + a^2 = c^2(x^2 - b^2)$

(c) $y^2 + b^2 = c^2(x^2 - a^2)$ (d) $y^2 - a^2 = c^2(x^2 + b^2)$

Solution Let the slopes of the two tangents to the hyperbola

$$x^2/a^2 - y^2/b^2 = 1 \text{ be } cm \text{ and } c/m$$

then the equation of the tangents are

$$y = cmx + \sqrt{a^2c^2m^2 - b^2} \quad (1)$$

and

$$my - cx = \sqrt{a^2c^2 - b^2m^2} \quad (2)$$

Squaring and subtracting (2) from (1) we get

$$(y - cmx)^2 - (my - cx)^2 = a^2c^2m^2 - b^2 - a^2c^2 + b^2m^2$$

$$\Rightarrow \quad (1 - m^2)(y^2 - c^2x^2) = -(1 - m^2)(a^2c^2 + b^2)$$

$$\Rightarrow \quad y^2 + b^2 = c^2(x^2 - a^2).$$

Question

If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, and $S(x_4, y_4)$, then

(a) $x_1 + x_2 + x_3 + x_4 = 0$ (b) $y_1 + y_2 + y_3 + y_4 = 2$

(c) $x_1x_2x_3x_4 = 2c^4$ (d) $y_1y_2y_3y_4 = 2c^4$

Solution

(a). Since $y = \frac{c^2}{x}$ and $x^2 + y^2 = a^2$

$\Rightarrow \quad x^2 + \frac{c^4}{x^2} = a^2$

$\Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$

This has four roots say x_1, x_2, x_3, x_4

$\therefore \quad x_1 + x_2 + x_3 + x_4 = 0$

Question

If a, b, c are in A.P., a, x, b are in G.P. and b, y, c are in G.P., the point (x, y) lies on

(a) a straight line
(b) a circle
(c) an ellipse
(d) a hyperbola

Ans. (b)

Solution We have $2b = a + c$, $x^2 = ab$, $y^2 = bc$ so that $x^2 + y^2 = b(a + c) = 2b^2$ which is a circle.

Question

A pont on the ellipse $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at a distance equal to the mean of the lengths of the semi-major axis and semi-minor axis from the centre is

(a) $\left(\frac{2\sqrt{91}}{7}, \frac{3\sqrt{105}}{14}\right)$
(b) $\left(-\frac{2\sqrt{91}}{7}, -\frac{3\sqrt{105}}{14}\right)$
(c) $\left(\frac{2\sqrt{105}}{7}, -\frac{3\sqrt{91}}{14}\right)$
(d) $\left(-\frac{2\sqrt{105}}{7}, \frac{3\sqrt{91}}{14}\right)$

Solution

(a, b, c, d). Let the point is $(4\cos\theta, 3\sin\theta)$

According to the question,

$$(4\cos\theta)^2 + (3\sin\theta)^2 = \left(\frac{4+3}{2}\right)^2 \quad \text{...(1)}$$

From (1), $16 - 7\sin^2\theta = \frac{49}{4} \Rightarrow \sin^2\theta = \frac{15}{28}$

$\therefore \quad \sin\theta = \pm\frac{1}{2}\sqrt{\frac{15}{7}} = \pm\frac{\sqrt{105}}{14}$

Similarly, $\cos\theta = \pm\frac{\sqrt{91}}{14}$

Question

If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin then the eccentricity of the hyperbola is

(a) $\sqrt{386}/12$ (b) $\sqrt{386}/13$ (c) $\sqrt{386}/25$ (d) $\sqrt{386}/38$

Ans. (a)

Solution Let $S(5, 12)$ and $S'(24, 7)$ be the two foci and $P(0, 0)$ be a point on the conic

then $SP = \sqrt{25+144} = \sqrt{169} = 13$; $S'P = \sqrt{(24)^2 + 7^2} = \sqrt{625} = 25$

and $SS' = \sqrt{(24-5)^2 + (7-12)^2} = \sqrt{19^2 + 5^2} = \sqrt{386}$

since the conic is a hyperbola, $S'P - SP = 2a$, the length of transverse axis and $SS' = 2ae$, e being the eccentricity.

$$\Rightarrow \quad e = \frac{SS'}{S'P + SP} = \frac{\sqrt{386}}{12}.$$

Question

C is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangents at any point P on this hyperbola meets the straight lines $bx - ay = 0$ and $bx + ay = 0$ in the points Q and R respectively. Then $CQ \cdot CR =$

(a) $a^2 + b^2$

(b) $a^2 - b^2$

(c) $\frac{1}{a^2} + \frac{1}{b^2}$

(d) $\frac{1}{a^2} - \frac{1}{b^2}$

Solution

(a). The coordinates of the point P are $(a \sec\theta, b \tan\theta)$

Tangent at P is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$

It meets $bx - ay = 0$ i.e., $\frac{x}{a} = \frac{y}{b}$ in Q

$\therefore \quad Q$ is $\left(\frac{a}{\sec\theta - \tan\theta} \cdot \frac{-b}{\sec\theta - \tan\theta}\right)$

It meets $bx + ay = 0$ i.e., $\frac{x}{a} = -\frac{y}{b}$ in R.

$\therefore \quad R$ is $\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$

$$\therefore \quad CQ \cdot CR = \frac{\sqrt{a^2 + b^2}}{(\sec\theta - \tan\theta)} \cdot \frac{\sqrt{a^2 + b^2}}{(\sec\theta + \tan\theta)}$$

$$= a^2 + b^2, \quad \{\because \sec^2\theta - \tan^2\theta = 1\}$$

Question

If P is a point on the rectangular hyperbola $x^2 - y^2 = a^2$, C is its centre and S, S' are the two foci, then $SP. S'P =$

(a) 2 (b) $(CP)^2$ (c) $(CS)^2$ (d) $(SS')^2$

Ans. (b)

Solution Let the coordinates of P be (x, y)

The coordinates of the centre C are $(0, 0)$

The eccentricity of the hyperbola is $\sqrt{1+\frac{a^2}{a^2}} = \sqrt{2}$

So the coordinates of the foci are $S(a\sqrt{2}, 0)$ and $S'(-a\sqrt{2}, 0)$.

Equation of the corresponding directrices are $x = a/\sqrt{2}$ and S $= - a/\sqrt{2}$.

By definition of the hyperbola

$$SP = e \text{ (distance of } P \text{ from } x = a/\sqrt{2})$$
$$= \sqrt{2}\,|x - a/\sqrt{2}|$$

Similarly $S'P = \sqrt{2}\,|x + a/\sqrt{2}|$

So that $SP . S'.P = 2\,|x^2 - a^2/2| = 2x^2 - a^2 = x^2 + y^2 = (CP)^2$

($\because$ P lies on the hyperbola $x^2 - y^2 = a^2$)

Question

Let PQ be a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If O be the centre of the hyperbola and OPQ is an equilateral triangle, then the eccentricity e is

(a) $> \sqrt{3}$ (b) >2

(c) $> \frac{2}{\sqrt{3}}$ (d) none of these

Solution

(c). Let P be (α, β). Then $PQ = 2\beta$ and $OP = \sqrt{\alpha^2 + \beta^2}$

Since OPQ is an equilateral triangle

$\therefore \quad OP = PQ \Rightarrow \alpha^2 + \beta^2 = 4\beta^2$

$$\Rightarrow \alpha^2 = 3\beta^2 \Rightarrow \alpha = \pm\sqrt{3}\,\beta$$

Since (α, β) is on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} = 1 \Rightarrow \frac{3\beta^2}{a^2} - \frac{\beta^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{3}{a^2} - \frac{1}{b^2} = \frac{1}{\beta^2} > 0$$

$$\Rightarrow \quad \frac{b^2}{a^2} > \frac{1}{3} \Rightarrow e^2 - 1 > \frac{1}{3}$$

$$\Rightarrow \quad e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}.$$

Question

The equation of a line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of its asymptotes is $3x - 4y - 6 = 0$, the equation of the other asymptote is

(a) $4x - 3y + 17 = 0$ (b) $-4x - 3y + 17 = 0$

(c) $-4x + 3y + 1 = 0$ (d) $4x + 3y + 17 = 0$

Solution

(d). We know that asymptotes of rectangular hyperbola are mutually perpendicular, thus other asymptote should be $4x + 3y + \lambda = 0$. Intersection point of asymptotes is also the centre of the hyperbola. Hence intersection point of $4x + 3y + \lambda = 0$ and $3x - 4y - 6 = 0$ should lie on the line $x - y - 1 = 0$. Using it λ can be easily obtained.

Question

The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. The value of b^2 is

(a) 9 (b) 1

(c) 5 (d) 7

Solution

(d). The equation of hyperbola is $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

Here, $a = \sqrt{\frac{144}{25}}$, $b = \sqrt{\frac{81}{25}}$, $e = \sqrt{1 + \frac{81}{144}} = \frac{15}{12} = \frac{5}{4}$

$\therefore$ Foci $= (\pm 3, 0)$.

Also, focus of ellipse $= (3, 0) \Rightarrow e = \frac{3}{4}$

$\therefore \; b^2 = 16\left(1 - \frac{9}{16}\right) = 7$

Question

If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$

(b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$

(d) $9x^2 - 8y^2 + 18x + 9 = 0$

Solution

(b). $x = 9$ meets the hyperbola $x^2 - y^2 = 9$ at $(9, 6\sqrt{2})$ and $(9, -6\sqrt{2})$. The equation of the tangents to the hyperbola at these points are $3x - 2\sqrt{2}y - 3 = 0$ and $3x + 2\sqrt{2}y - 3 = 0$.

Joint equation of the two tangents is therefore

$$(3x - 2\sqrt{2}y - 3)(3x + 2\sqrt{2}y - 3) = 0$$

$$\Rightarrow \quad (3x - 3)^2 - (2\sqrt{2}\,y)^2 = 0$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

Question

The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is

(a) $3x - 4y = 4$

(b) $3x - 4y + 4 = 0$

(c) $4x - 4y = 3$

(d) $3x - 4y = 2$

Solution

(a). Let the mid point be (h, k). Equation of a chord whose mid point is (h, k) would be $T = S_1$

or $3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$

$\Rightarrow x(3h + 2) - y(2k + 3) - (2h + 3k) - 3h^2 + 2k^2 = 0$

Its slope is $\dfrac{3h+2}{2k+3} = 2$ (given) $\Rightarrow 3h = 4k + 4$

$\therefore$ Required locus is $3x - 4y = 4$.

Question

If S and S' be the foci, C the centre and P be any point on a rectangular hyperbola, then $SP \cdot S'P$ is equal to

(a) CP (b) $(CP)^2$

(c) $(CP)^4$ (d) $(CP)^3$

Solution

(b). Rectangular hyperbola is $x^2 - y^2 = a^2$ ...(1)

$e = \sqrt{2}$, $C(0, 0)$, $S(\sqrt{2}a, 0)$, $S'(-\sqrt{2}a, 0)$

Let $P(\alpha, \beta)$ by any point

P lies on (1)

$\therefore \quad \alpha^2 - \beta^2 = a^2$ ...(2)

Now $SP^2 = (\sqrt{2}a - \alpha)^2 + (0 - \beta)^2$

$= 2a^2 + \alpha^2 + \beta^2 - 2\sqrt{2}a\alpha$

$S'P^2 = (-\sqrt{2}a - \alpha)^2 + (0 - \beta)^2$

$= 2a^2 + \alpha^2 + \beta^2 + 2\sqrt{2}\,a\alpha$

Now $SP^2 \cdot S'P^2 = (2a^2 + \alpha^2 + \beta^2)^2 - 8a^2\alpha^2$

$= 4a^4 + 4a^2(\alpha^2 + \beta^2) + (\alpha^2 + \beta^2)^2 - 8a^2\alpha^2$

$= 4a^2(a^2 - 2\alpha^2) + 4a^2(\alpha^2 + \beta^2) + (\alpha^2 + \beta^2)^2$

$= 4a^2(\alpha^2 - \beta^2 - 2\alpha^2) + 4a^2(\alpha^2 + \beta^2) + (\alpha^2 + \beta^2)^2$

$= -4a^2(\alpha^2 + \beta^2) + 4a^2(\alpha^2 + \beta^2) + (\alpha^2 + \beta^2)^2$

$= (\alpha^2 + \beta^2)^2 = (CP^2)^2 = CP^4$

$\therefore \quad SP \cdot S'P = CP^2$

Question

If AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔOAB is an equilateral triangle, O being the origin, then the eccentricity of the hyperbola satisfies

(a) $e > \sqrt{3}$

(b) $1 < e < \frac{2}{\sqrt{3}}$

(c) $e = \frac{2}{\sqrt{3}}$

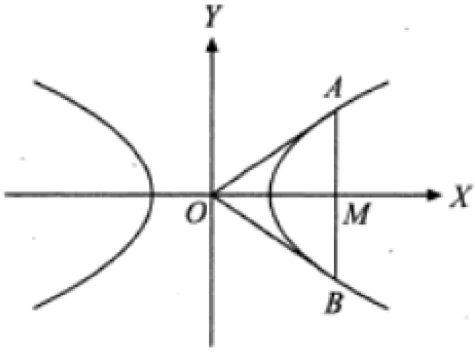
(d) $e > \frac{2}{\sqrt{3}}$

Solution

(d). Let the length of the double ordinate be $2l$.

$\therefore$ $AB = 2l$ and $AM = BM = l$

Clearly ordinate of point A is l



The abscissa of the point A is given by

$$\frac{x^2}{a^2} - \frac{l^2}{b^2} = 1 \Rightarrow x = \frac{a\sqrt{b^2 + l^2}}{b}$$

$\therefore$ A is $\left(\frac{a\sqrt{b^2 + l^2}}{b}, l\right)$

Since ΔOAB is equilateral triangle, therefore

$$OA = AB = OB = 2l$$

Also, $OM^2 + AM^2 = OA^2$

$$\Rightarrow \frac{a^2(b^2 + l^2)}{b^2} + l^2 = 4l^2$$

$$\therefore \quad l^2 = \frac{a^2b^2}{3b^2 - a^2}$$

Since $l^2 > 0$

$$\therefore \quad \frac{a^2b^2}{3b^2 - a^2} > 0 \Rightarrow 3b^2 - a^2 > 0$$

$$\Rightarrow \quad 3a^2(e^2 - 1) - a^2 > 0$$

$$\Rightarrow \quad e > \frac{2}{\sqrt{3}}$$

Question

For the curve $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$ which of the following is true:

(a) an hyperbola with eccentricity $\sqrt{3}$

(b) an hyperbola with directrix $2x + y - 1 = 0$

(c) an hyperbola with focus $(1, 2)$

(d) All of these

Solution

(d). Given $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

$$\Rightarrow 7x^2 - 2y^2 = -[2 \times 2 \times 3xy - 2x + 14y - 22]$$

$$= -3(2x + y - 1)^2 + 12x^2 + 3y^2 + 25 - 10x - 20y$$

$$\Rightarrow 7x^2 - 2y^2 - 12x^2 - 3y^2 - 25 = -3(2x + y - 1)^2 - 10x - 20y$$

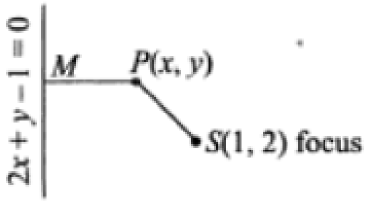
$$\Rightarrow -5[x^2 + y^2 - 2x - 4y + 5] = -3(2x + y - 1)^2$$

$$\Rightarrow (x - 1)^2 + (y - 2)^2 = \frac{3}{5}(2x + y - 1)^2$$

$$\Rightarrow (x - 1)^2 + (y - 2)^2 = 3\left(\frac{2x + y - 1}{\sqrt{2^2 + 1}}\right)^2$$

$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{3}\left(\frac{2x + y - 1}{\sqrt{5}}\right)$$

$$\Rightarrow \frac{PS}{PM} = \sqrt{3} > 1$$



Therefore, the given equation represents a hyperbola with eccentricity $\sqrt{3}$

Directrix is $2x + y - 1 = 0$ and

focus is $(1, 2)$.

Question

The difference between the length $2a$ of the transverse axis of a hyperbola of eccentricity e and the length of its latus rectum is

(a) $2a|3 - e^2|$ (b) $2a|2 - e^2|$

(c) $2a(e^2 - 1)$ (d) $a(2e^2 - 1)$

Solution

(b). Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Length of transverse axis is $2a$ and

Length of latus rectum is $\frac{2b^2}{a}$

Now, difference $= \left|2a - \frac{2b^2}{a}\right| = \frac{2}{a}|2a^2 - a^2e^2|$

$\therefore$ Difference $= 2a|2 - e^2|$.

Question

If a variable line $x\cos\alpha + y\sin\alpha = p$ which is a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(b > a)$ subtends a right angle at the centre of the hyperbola, then it always touches a fixed circle whose radius is

(a) $\frac{ab}{\sqrt{a^2+b^2}}$ (b) $\frac{ab}{\sqrt{b^2-a^2}}$

(c) $\frac{ab}{\sqrt{a^2-b^2}}$ (d) none of these

Solution

(b). Since $x\cos\alpha + y\sin\alpha = p$ subtends a right angle at the centre (0, 0), therefore

making equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ homogeneous with the help of $x\cos\alpha + y\sin\alpha = p$

we get $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$

i.e. $x^2\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) + y^2\left(-\frac{1}{b^2} - \frac{\sin^2\alpha}{p^2}\right) + \frac{-2\sin\alpha\cos\alpha\, xy}{p^2} = 0$

co-eff. of x^2 + co-eff. of $y^2 = 0$

$$\Rightarrow \frac{1}{a^2} - \frac{\cos^2\alpha}{p^2} - \frac{1}{b^2} - \frac{\sin^2\alpha}{p^2} = 0$$

$$\Rightarrow \frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2} \Rightarrow p = \frac{ab}{\sqrt{b^2-a^2}}.$$

Since p is also the length of the perpendicular from (0, 0) to the line $x\cos\alpha + y\sin\alpha = p$

$\therefore$ Radius of the circle $= p = \dfrac{ab}{\sqrt{b^2 - a^2}}$

Question

If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$

(b) $9x^2 - 8y^2 + 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$

(d) $9x^2 - 8y^2 + 18x + 9 = 0$

Solution

(b). Chord of contact of (x_1, y_1) is $xx_1 - yy_1 = 9$

Given chord of contact is $x = 9$. So $x_1 = 1, y_1 = 0$

Pair of tangents from $(1, 0)$ is

$(x^2 - y^2 - 9)(-8) = (x - 9)^2$

i.e. $9x^2 - 8y^2 - 18x + 9 = 0$

Question

If e is the eccentricity of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and θ be the angle between the asymptotes then $\sec \theta/2$ equals

(a) e^2

(b) $\dfrac{1}{e}$

(c) $2e$

(d) e

Solution

(d). Equation of asymptotes to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by

$y=-\frac{b}{a}x$ and $y=\frac{b}{a}\overline{x}$

$\therefore \quad m_1=-b/a$ and $m_2=\frac{b}{a}$

Similarly $y=\frac{bx}{a}$

$\therefore \quad m_2=b/a$

Now $\theta=2\tan^{-1}(b/a)$

$\therefore \quad \tan\theta/2=b/a \Rightarrow \tan^2\theta/2=\frac{b^2}{a^2}=e^2-1$

$\therefore \quad \sec^2\theta/2=e^2$ or $\sec\theta/2=e$

Question

If two tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are drawn such that the product of their gradients is c^2, then they intersect at the curve

(a) $y^2+b^2=c^2(x^2-a^2)$ (b) $ax^2+by^2=c^2$

(c) $y^2+b^2=c^2(x^2+a^2)$ (d) $y^2-b^2=c^2(x^2-a^2)$

Solution

(a). Let the tangents meet at the point (h, k), then equation of tangents drawn from (h, k) is given by

$SS_1=T^2$

$$\Rightarrow \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}-1\right)=\left(\frac{xh}{a^2}-\frac{yk}{b^2}-1\right)^2$$

$$\Rightarrow \quad x^2\left(\frac{h^2}{a^4}-\frac{k^2}{a^2b^2}-\frac{1}{a^2}-\frac{h^2}{a^4}\right)$$

$$-y^2\left(\frac{h^2}{a^2b^2}-\frac{k^2}{b^4}-\frac{1}{b^2}+\right. \qquad = 0$$

But $m_1m_2 = \left(\dfrac{\text{coefficient of } x^2}{\text{coefficient of } y^2}\right)$

$$\Rightarrow \quad m_1m_2 = \frac{\dfrac{k^2}{a^2b^2}+\dfrac{1}{a^2}}{\dfrac{h^2}{a^2b^2}-\dfrac{1}{b^2}} = c^2 \Rightarrow \frac{(k^2+b^2)}{(h^2-a^2)} = c^2$$

$\therefore \quad y^2 + b^2 = c^2(x^2 - a^2)$

Question on Hyperbola and Locus

If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Then the locus of the middle points of these chords is the curve:

(a) $y^2 = (x - a)x^3$ (b) $y^2(x - a) = x^3$

(c) $x^2(x - a) = y^3$ (d) none of these

Solution

(b). Equation of chord of hyperbola $x^2 - y^2 = a^2$ with mid-point as (h, k) is given by

$$xh - yk = h^2 - k^2 \text{ or } y = \frac{h}{k}x - \frac{(h^2-k^2)}{k}$$

This will touch the parabiola

$$y^2 = 4ax \text{ if } -\left(\frac{h^2-k^2}{k}\right) = \frac{a}{h/k}$$

$\Rightarrow \quad k^2(h - a) = h^3$

$\therefore \quad$ The locus is $y^2 (x - a) = x^3$.

Question

The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the directrix in F. If PF subtends an angle θ at the corresponding focus, then θ equals

(a) $\pi/4$ (b) $\pi/2$

(c) $3\pi/4$ (d) π

Solution

(b). Let the directrix be $x = a/e$ and focus be $S(ae, 0)$. Let $P(a \sec\theta, b\tan\theta)$ be any point on the curve. Equation of tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. Let F be the intersection point of tangent and the directrix, then $F = \left(\frac{a}{e}, \frac{b(\sec\theta - e)}{e\tan\theta}\right)$

$$\Rightarrow \quad m_{SF} = \frac{b(\sec\theta - e)}{-e\tan\theta\,(a^2 - 1)},\ m_{PS} = \frac{b\tan\theta}{a(\sec\theta - e)}$$

$$\Rightarrow \quad m_{SF} \cdot m_{PS} = -1.$$

Question on Hyperbola and Parabola combined

A parabola is drawn with its vertex at (0, – 3), the axis of symmetry along the conjugate axis of the hyperbola $\frac{x^2}{49} - \frac{y^2}{9} = 1$ and passing through the two foci of the hyperbola. The coordinates of the focus of the parabola are

(a) $\left(0, \frac{11}{6}\right)$ (b) $\left(0, -\frac{11}{6}\right)$

(c) $\left(0, \frac{11}{12}\right)$ (d) $\left(0, -\frac{11}{12}\right)$

Solution

(a). Eqn. of hyperbola is $\frac{x^2}{49} - \frac{y^2}{9} = 1$

Its conjugate axis is y-axis.

Also, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{49}} = \frac{\sqrt{58}}{7}$

$\therefore$ Foci of hyperbola is $(\pm ae, 0)$, i.e. $(\pm\sqrt{58}, 0)$.

Now equation of parabola with vertex at (0, – 3) and axis along y-axis is $x^2 = l(y + 3)$

It passes through $(\pm\sqrt{58}, 0)$.

$\therefore$ $58 = l(0 + 3) \Rightarrow l = \frac{58}{3}$

$\therefore$ Parabola is $x^2 = \frac{58}{3}(y + 3)$

Its focus is $\left(0, -3 + \frac{58}{4.3}\right)$ or $\left(0, \frac{11}{6}\right)$.

Question

For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ which of the following remains constant as α varies

(a) eccentricity
(b) directrix
(c) Abscissae of vertices
(d) Abscissae of foccii

Ans. (d)

Solution Abscissae of the foci = $\pm ae$ where $ae = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

:-{D

To recall standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}$ $(n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1}$ $(n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a}$ $(a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\lvert\tan\frac{x}{2}\rvert$	$\operatorname{cosech} x$	$\ln\lvert\tanh\frac{x}{2}\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$ $(a>0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\|$ $(0<\|x\|<a)$
		$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|$ $(\|x\|>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$ $(-a<x<a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\|\frac{x+\sqrt{a^2+x^2}}{a}\right\|$ $(a>0)$
		$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\|\frac{x+\sqrt{x^2-a^2}}{a}\right\|$ $(x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
		$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

Some series Expansions -

$$\frac{\pi}{2}=\left(\frac{2}{1}\frac{2}{3}\right)\left(\frac{4}{3}\frac{4}{5}\right)\left(\frac{6}{5}\frac{6}{7}\right)\left(\frac{8}{7}\frac{8}{9}\right)\ldots$$

$$\pi=\frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\frac{4}{13}-\cdots$$

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$$

$$\pi=\sqrt{12}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots\right)$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$$

Solve a series problem

If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ upto $\infty = \frac{\pi^2}{6}$, then value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ up to ∞ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$

Ans. (c)

Solution We have $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ upto ∞

$$= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\cdots \text{ upto } \infty$$

$$-\frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$= \frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\ldots\infty=\frac{\pi^2}{12}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\infty=\frac{\pi^2}{24}$$

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\frac{x^4}{9!}-\frac{x^5}{11!}+\ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \le x < 1)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} \cdots + \frac{2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots + \frac{2\left(2^{2n-1}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} - \cdots \qquad 0 < |x| < \pi$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\sec x = 1 + \frac{x^2}{\lfloor 2} + \frac{5x^4}{\lfloor 4} + \cdots$$

$$\log(\cos x) = -\frac{x^2}{\lfloor 2} - \frac{2x^4}{\lfloor 4} - \cdots$$

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \quad |x| < 1$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$= \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right) \quad |x| < 1$$

$$\tan^{-1} x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & |x| < 1 \\ \pm\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & \begin{cases} + \text{ if } x \geq 1 \\ - \text{ if } x \leq -1 \end{cases} \end{cases}$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$= \frac{\pi}{2} - \left(\frac{1}{x} + \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots\right) \quad |x| > 1$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

$$= \frac{1}{x} + \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots \quad |x| > 1$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$= \begin{cases} \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) & |x| < 1 \\ p\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots & \begin{cases} p = 0 \text{ if } x \geq 1 \\ p = 1 \text{ if } x \leq -1 \end{cases} \end{cases}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right]$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1} \quad (x > 0)$$

$$\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x-1}{x}\right)^n \quad (x > \frac{1}{2})$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}(x-1)^n \quad (0 < x \le 2)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}x^n \quad (|x| < 1)$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\infty \, (-1 \le x < 1)$$

$$\log_e(1+x) - \log_e(1-x) =$$

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty\right)(-1 < x < 1)$$

$$\log_e\left(1+\frac{1}{n}\right) = \log_e \frac{n+1}{n} = 2 \quad \left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\infty\right]$$

$$\log_e (1 + x) + \log_e (1 - x) = \log_e(1-x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \ldots\infty\right)(-1 < x < 1)$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \quad = \frac{1}{1\cdot 2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$$

Important Results

(i) (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + \tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1 + \cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x} dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \text{cosec}^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\text{cosec}^n x}{\sec^n x + \text{cosec}^n x} dx$ where, $n \in R$

(ii) $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \frac{\pi}{4}$

(iii) (a) $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$

(b) $\int_0^{\pi/2} \log \tan x \, dx = \int_0^{\pi/2} \log \cot x \, dx = 0$

(c) $\int_0^{\pi/2} \log \sec x \, dx = \int_0^{\pi/2} \log \text{cosec} x \, dx = \frac{\pi}{2} \log 2$

(iv) (a) $\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$

(b) $\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$

(c) $\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^n + 1}$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)+C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)+C$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+C$$

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Good Luck to you for your Preparations, References, and Exams

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