# Spoon Feeding Parabola

**Simplified Knowledge Management Classes Bangalore**

My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams ( such as IMO [ International Mathematics Olympiad ], IPhO [ International Physics Olympiad ], IChO [ International Chemistry Olympiad ] ), IGCSE ( IB ), CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25 th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education ( HBCSE ) Physics Olympics camp BARC Campus.

**I am Life Member of …**

- IAPT ( Indian Association of Physics Teachers )
- IPA ( Indian Physics Association )
- AMTI ( Association of Mathematics Teachers of India )
- National Human Rights Association
- Men's Rights Movement ( India and International )
- MGTOW Movement ( India and International )

**And also of**

IACT ( Indian Association of Chemistry Teachers )

**The selection for National Camp ( for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy ) happens in the following steps ….**

1 ) **NSEP** ( National Standard Exam in Physics ) and **NSEC** ( National Standard Exam in Chemistry ) held around 24 rth November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2 ) **INPhO** ( Indian National Physics Olympiad ) and **INChO** ( Indian National Chemistry Olympiad ). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3 ) The Top 35 students of each subject are invited at HBCSE ( Homi Bhabha Center for Science Education ) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of "Good Books". Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.

**There are 3 kinds of Text Books**

- The thin Books - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to "Cram" quickly and pass somehow find the thin books "good" as they have to read less !!

**- The Thick Books - Most students do not like these, as they want to read as less as possible. Average students are "busy" with many other things and have no time to read all these.**

- The Average sized Books - Good students do not get all details in any one book. Most bad students do not want to read books of "this much thickness" also !!

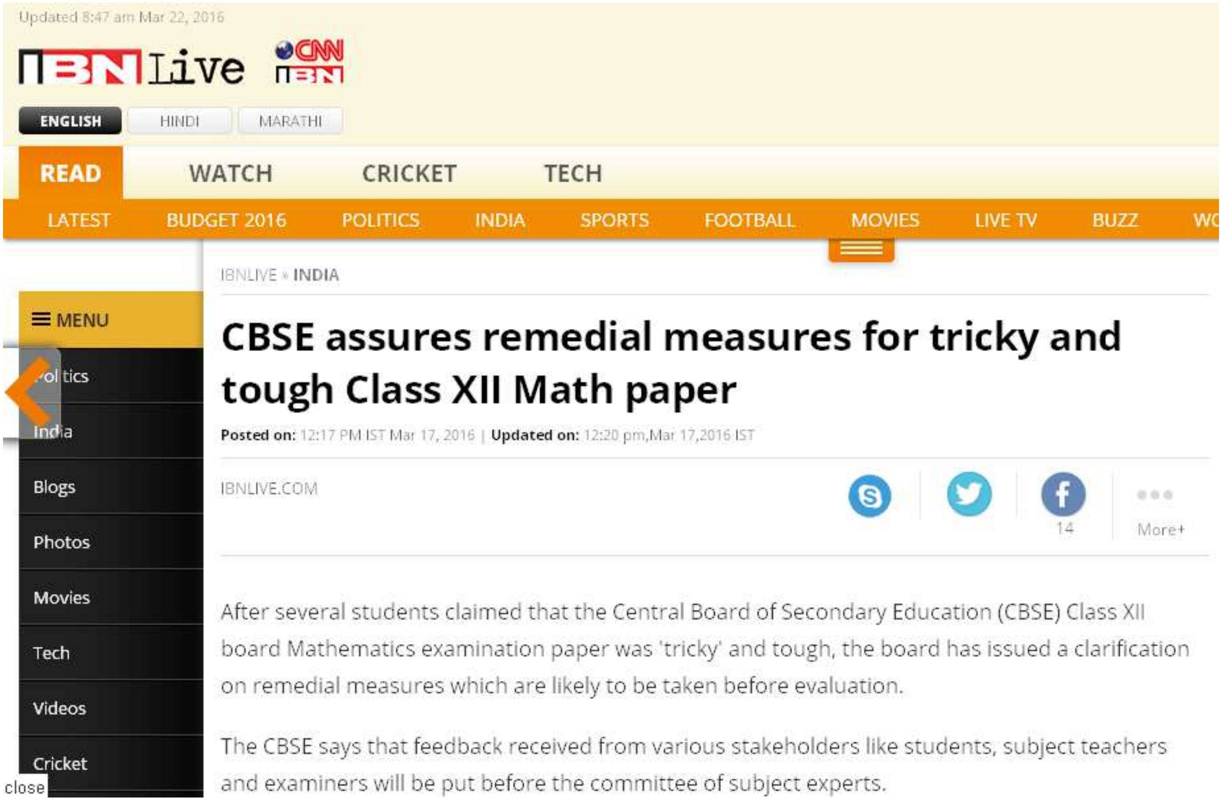
**We know there can be no shoe that's fits in all.**

Printed books are not e-Books! Can't be downloaded and kept in hard-disc for reading "later" ………

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good "Reference Material". I sincerely wish that all find this "very useful".

Students who do not practice lots of problems, do not do well. The rules of "doing well" had never changed …. Will never change !

After 2016 CBSE Mathematics exam, lots of students complained that the paper was tough!



Updated 8:47 am Mar 22, 2016

ENGLISH HINDI MARATHI

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IBNLIVE » INDIA

# CBSE assures remedial measures for tricky and tough Class XII Math paper

**Posted on:** 12:17 PM IST Mar 17, 2016 | **Updated on:** 12:20 pm,Mar 17,2016 IST

IBNLIVE.COM

After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.

On 21 st May 2016 the CBSE standard 12 result was declared. I loved the headline

**INDIATODAY.IN** NEW DELHI, MAY 21, 2016 | UPDATED 16:40 IST

# CBSE Class 12 Results out: No leniency in Maths paper, high paper standard to be maintained in future

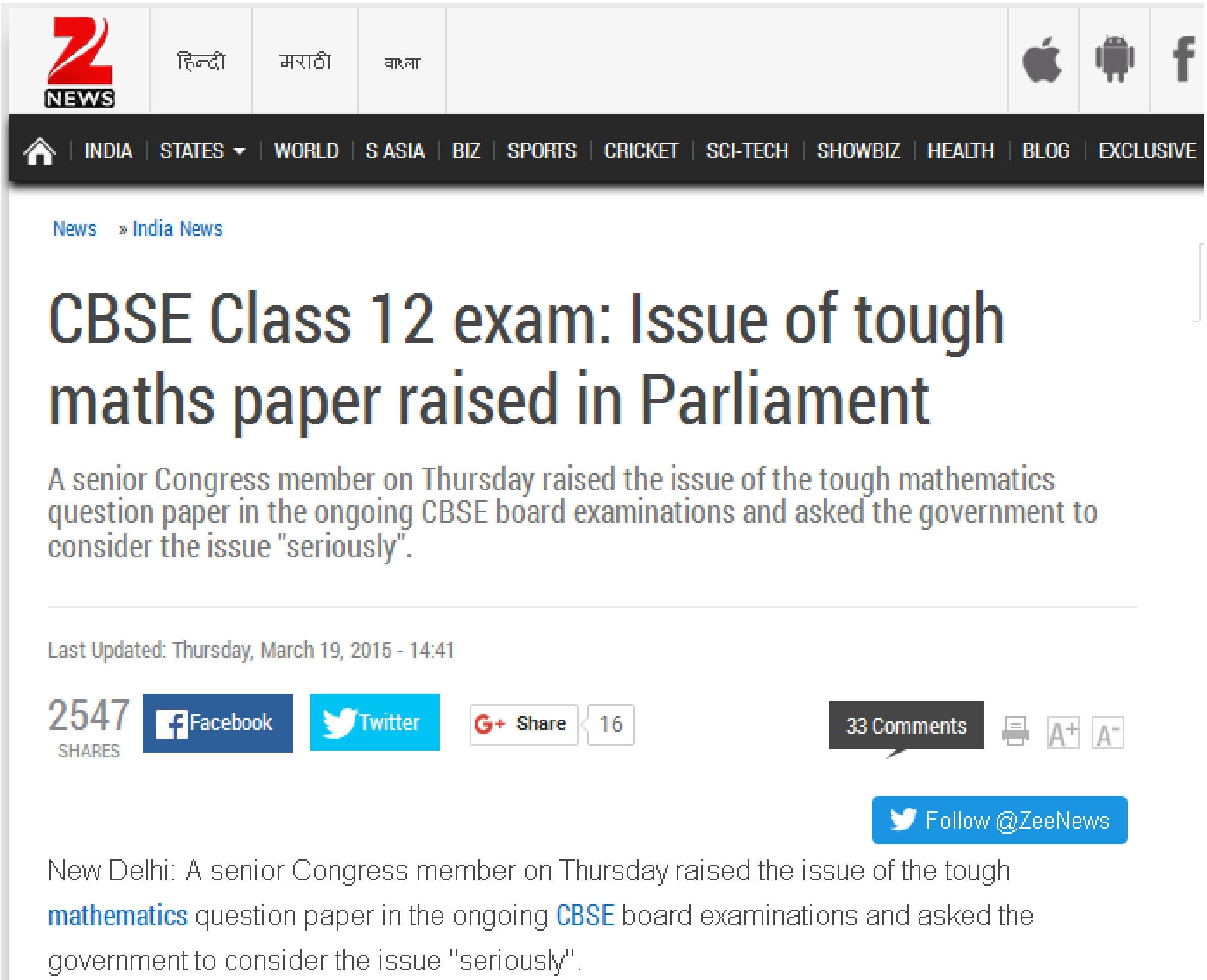
The CBSE Class 12 Mathematics board exam on March 14 reduced many students to tears as they found the paper quite lengthy and tough and many couldn't finish it on time. The results show an overall lowering of marks received in the Maths paper.

**RELATED STORIES**

- CBSE Board result 2016 declared! Thiruvanathpuram obtains the highest part percentage, check how your region scored
- Meet CBSE topper Sukriti Gupta: Check her percentage here!
- CBSE Class 12 Boards 2016: Results announced ahead of time!
- CBSE results declared at www.cbse.nic.in: Steps to check online
- Exclusive! CBSE declares Class 12 Results at www.cbseresults.nic.in and cbse.nic.in

The CBSE (Central Board of Secondary Education) Class 12 Board exam results have been announced today, i.e on May 21, around 10:30 am ahead of time. Students may check their scores at the official website, *www.cbseresults.nic.in*. (**Read: CBSE Class 12 Boards 2016: Results announced ahead of time! Check your score at cbseresults.nic.in**)

In 2015 also the same complain was there by many students



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News » India News

# CBSE Class 12 exam: Issue of tough maths paper raised in Parliament

A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

Last Updated: Thursday, March 19, 2015 - 14:41

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New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

So we see that by raising frivolous requests, even upto parliament, actually does not help. Many times requests from several quarters have been put to CBSE, or Parliament etc for easy Math Paper. These kinds of requests actually can-not be entertained, never will be.

In March 2016, students of Karnataka PU-II also complained the same, regarding standard 12 ( PU-II Mathematics Exam ). Even though the Math Paper was identical to previous year, most students had not even solved the 2015 Question Paper.

Friday, March 25, 2016 - 13:28

The NEWS Minute

HOME NEWS ANDHRA KARNATAKA KERALA TAMIL NADU TELANGANA CULTURE MEDIA BLOG

Exams

Online petition for lenient evaluation of K'taka II PU math paper gets over 8000 supporters

The campaign, which was launched on Monday, has garnered over 8000 supporters

TNM Staff | Wednesday, March 16, 2016 - 09:32

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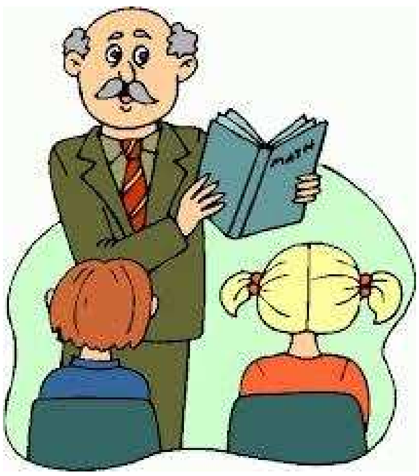
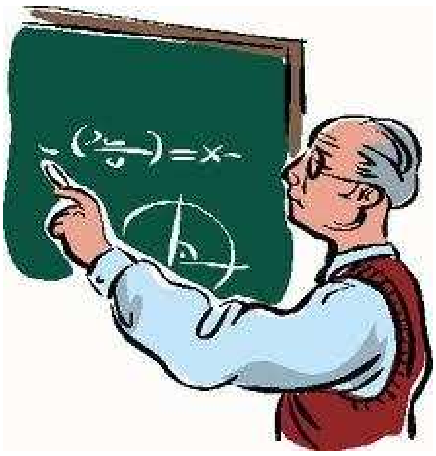
Following a "very tough" math paper that left many II PU students in tears, Saket Ravindran a student launched an online campaign demanding lenient evaluation.

These complains are not new. In fact since last 40 years, ( since my childhood ), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

**No one can help those who are not studying, or practicing.**





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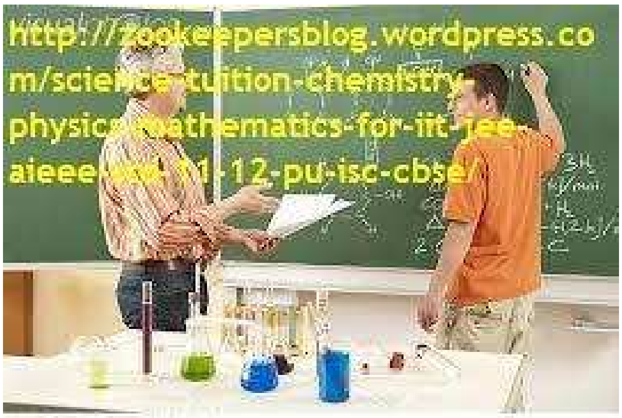


### A very polite request :

I wish these e-Books are read only by Boys and Men. Girls and Women, better read something else; learn from somewhere else.

## Preface

We all know that in the species "Homo Sapiens ", males are bigger than females. The reasons are explained in standard 10, or 11 ( high school ) Biology texts. **This shapes or size, influences all of our culture**. Before we recall / understand the reasons once again, let us see some random examples of the influence

Random - 1

If there is a Road rage, then who all fight ? ( generally ? ). Imagine two cars driven by adult drivers. Each car has a woman of similar age as that of the Man. The cars " touch " or " some issue happens". Who all comes out and fights ? Who all are most probable to drive the cars ?

( Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win )

Random - 2

Heavy metal music artists are all Men. Metallica, Black Sabbath, Motley Crue, Megadeth, Motorhead, AC/DC, Deep Purple, Slayer, Guns & Roses, Led Zeppelin, Aerosmith ….. the list can be in thousands. All these are grown-up Boys, known as Men.

( Men strive for perfection. Men are eager to excel. Men work hard. Men want to win. )

Random - 3

Apart from Marie Curie, only one more woman got Nobel Prize in Physics. ( Maria Goeppert Mayer - 1963 ). So, … almost all are men.

( Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women. )

Random - 4

The best Tabla Players are all Men.

( Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women. )

Random - 5

History is all about, which Kings ruled. Kings, their men, and Soldiers went for wars. History is all about wars, fights, and killings by men.

**Boys start fighting from school days. Girls do not fight like this**

( Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win. )

Random - 6

The highest award in Mathematics, the " Fields Medal " is around since decades. Till date only one woman could get that. ( Maryam Mirzakhani - 2014 ). So, … almost all are men.

( Men want to excel. Men strive for perfection. Men want to win. Men work hard. Men do better than women. )

Random - 7

Actor is a gender neutral word. Could the movie like " Top Gun " be made with Female actors ? The best pilots, astronauts, Fighters are all Men.

Random - 8

In my childhood had seen a movie named " The Tower in Inferno ". In the movie when the tall tower is in fire, women were being saved first, as only one lift was working….


Many decades later another movie is made. A box office hit. " The Titanic ". In this also …. As the ship is sinking women are being saved. **Men are disposable**. Men may get their turn later…

Movies are not training programs. Movies do not teach people what to do, or not to do. Movies only reflect the prevalent culture. Men are disposable, is the culture in the society. Knowingly, unknowingly, the culture is depicted in Movies, Theaters, Stories, Poems, Rituals, etc. I or you can't write a story, or make a movie in which after a minor car accident the Male passengers keep seating in the back seat, while the both the women drivers come out of the car and start fighting very bitterly on the road. There has been no story in this world, or no movie made, where after an accident or calamity, Men are being helped for safety first, and women are told to wait.

Random - 9

Artists generally follow the prevalent culture of the Society. In paintings, sculptures, stories, poems, movies, cartoon, Caricatures, knowingly / unknowingly, “ the prevalent Reality “ is depicted. The opposite will not go well with people. If deliberately “ the opposite “ is shown then it may only become a special art, considered as a special mockery.



Random - 10

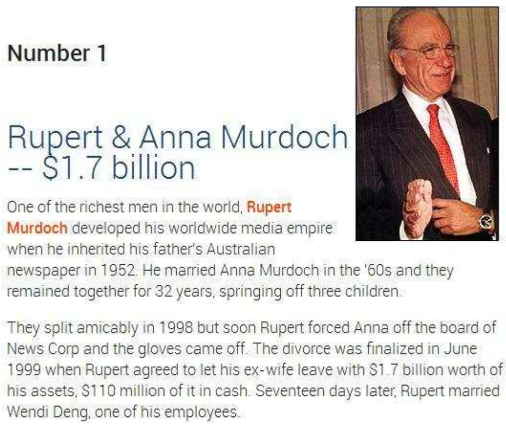
Men go to “girl / woman’s house” to marry / win, and bring her to his home. That is a sort of winning her. When a boy gets a “ Girl-Friend “, generally he and his friends consider that as an achievement. The boy who “ got / won “ a girl-friend feels proud. His male friends feel, jealous, competitive and envious. Millions of stories have been written on these themes. Lakhs of movies show this. Boys / Men go for “ bike race “, or say “ Car Race “, where the winner “ gets “ the most beautiful girl of the college.

( Men want to excel. Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win. )

Prithviraj Chauhan ‘ went ` to “ pickup “ or “ abduct “ or “ win “ or “ bring “ his love. There was a Hindi movie ( hit ) song … “ Pasand ho jaye, to ghar se utha laye “. It is not other way round. Girls do not go to Boy’s house or man’s house to marry. Nor the girls go in a gang to “ pick-up “ the boy / man and bring him to their home / place / den.

Random - 11

Rich people; often are very hard working. Successful business men, establish their business ( empire ), amass lot of wealth, with lot of difficulty. Lots of sacrifice, lots of hard work, gets into this. Rich people's wives had no contribution in this wealth creation. Women are smart, and successful upto the extent to choose the right/rich man to marry. So generally what happens in case of Divorces ? Search the net on " most costly divorces " and you will know. The women;( who had no contribution at all, in setting up the business / empire ), often gets in Billions, or several Millions in divorce settlements.



Number 1

Rupert & Anna Murdoch -- $1.7 billion

One of the richest men in the world, **Rupert Murdoch** developed his worldwide media empire when he inherited his father's Australian newspaper in 1952. He married Anna Murdoch in the '60s and they remained together for 32 years, springing off three children.

They split amicably in 1998 but soon Rupert forced Anna off the board of News Corp and the gloves came off. The divorce was finalized in June 1999 when Rupert agreed to let his ex-wife leave with $1.7 billion worth of his assets, $110 million of it in cash. Seventeen days later, Rupert married Wendi Deng, one of his employees.

Ted Danson & Casey Coates -- $30 million

Ted Danson's claim to fame is undoubtedly his decade-long stint as Sam Malone on NBC's celebrated sitcom Cheers . While he did other TV shows and movies, he will always be known as the bartender of that place where everybody knows your name. He met his future first bride Casey, a designer, in 1976 while doing Erhard Seminars Training.

Ten years his senior, she suffered a paralyzing stroke while giving birth to their first child in 1979. In order to nurse her back to health, Danson took a break from acting for six months. But after two children and 15 years of marriage, the infatuation fell to pieces. Danson had started seeing Whoopi Goldberg while filming the comedy, Made in America and this precipitated the 1992 divorce. Casey got $30 million for her trouble.

See https://zookeepersblog.wordpress.com/misandry-and-men-issues-a-short-summary-at-single-place/

See http://skmclasses.kinja.com/save-the-male-1761788732

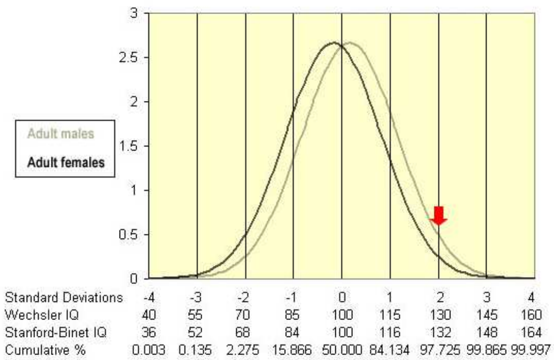
It was Boys and Men, who brought the girls / women home. The Laws are biased, completely favoring women. The men are paying for their own mistakes.

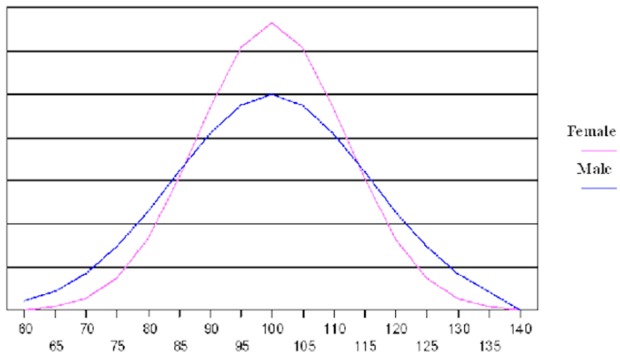
See https://zookeepersblog.wordpress.com/biased-laws/

( Man brings the Woman home. When she leaves, takes away her share of big fortune! )

Random - 12

A standardized test of Intelligence will never be possible. It never happened before, nor ever will happen in future; where the IQ test results will be acceptable by all. In the net there are thousands of charts which show that the intelligence scores of girls / women are lesser. Debates of Trillion words, does not improve performance of Girls.




I am not wasting a single second debating or discussing with anyone, on this. I am simply accepting ALL the results. IQ is only one of the variables which is required for success in life. Thousands of books have been written on “ Networking Skills “, EQ ( Emotional Quotient ), Drive, Dedication, Focus, “ Tenacity towards the end goal “ … etc. In each criteria, and in all together, women ( in general ) do far worse than men. Bangalore is known as “ ….. capital of India “. [ Fill in the blanks ]. The blanks are generally filled as “ Software Capital “, “ IT Capital “, “ Startup Capital “, etc. I am member in several startup eco-systems / groups. I have attended hundreds of meetings, regarding “ technology startups “, or “ idea startups “. These meetings have very few women. Starting up new companies are all “ Men’s Game “ / “ Men’s business “. Only in Divorce settlements women will take their goodies, due to Biased laws. There is no dedication, towards wealth creation, by women.

Random - 13

Many men, as fathers, very unfortunately treat their daughters as “ Princess “. Every “ non-performing “ woman / wife was “ princess daughter “ of some loving father. Pampering the girls, in name of “ equal opportunity “, or “ women empowerment “, have led to nothing.


See http://skmclasses.kinja.com/progressively-daughters-become-monsters-1764484338

See http://skmclasses.kinja.com/vivacious-vixens-1764483974

There can be thousands of more such random examples, where " Bigger Shape / size " of males have influenced our culture, our Society. **Let us recall the reasons**, that we already learned in standard 10 - 11, Biology text Books. In humans, women have a long gestation period, and also spends many years ( almost a decade ) to grow, nourish, and stabilize the child. ( Million years of habit ) Due to survival instinct Males want to inseminate. Boys and Men fight for the " facility ( of womb + care ) " the girl / woman may provide. Bigger size for males, has a winning advantage. Whoever wins, gets the " woman / facility ". The male who is of " Bigger Size ", has an advantage to win…. Leading to Natural selection over millions of years. In general " Bigger Males "; the " fighting instinct " in men; have led to wars, and solving tough problems ( Mathematics, Physics, Technology, startups of new businesses, Wealth creation, Unreasonable attempts to make things [ such as planes ], Hard work …. )

**So let us see the IIT-JEE results of girls**. Statistics of several years show that there are around 17, ( or less than 20 ) girls in top 1000 ranks, at all India level. Some people will yet not understand the performance, till it is said that … year after year we have around 980 boys in top 1000 ranks. Generally we see only 4 to 5 girls in top 500. In last 50 years not once any girl topped in IIT-JEE advanced. Forget about Single digit ranks, double digit ranks by girls have been extremely rare. It is all about " good boys ", " hard working ", " focused ", "**Bel-esprit** " boys.

**In 2015, Only 2.6% of total candidates who qualified are girls ( upto around 12,000 rank ). while 20% of the Boys, amongst all candidates qualified. The Total number of students who appeared for the exam were around 1.4 million for IIT-JEE main. Subsequently 1.2 lakh ( around 120 thousands ) appeared for IIT-JEE advanced.**

IIT-JEE results and analysis, of many years is given at https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/

In Bangalore it is rare to see a girl with rank better than 1000 in IIT-JEE advanced. We hardly see 6-7 boys with rank better than 1000. Hardly 2-3 boys get a rank better than 500.

See http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html

Thousands of people are exposing the heinous crimes that Motherly Women are doing, or Female Teachers are committing. See https://www.facebook.com/WomenCriminals/

Some Random Examples must be known by all


Mother Admits On Facebook to Sleeping with 15 Yr Old Son, They Have a Baby Together - Alwayzturntup

Sometimes it hard to believe w From Alwayzturntup

ALWAYZTURNTUP.ME

It is extremely unfortunate that the " woman empowerment " has created. This is the kind of society and women we have now. I and many other sensible Men hate such women. Be away from such women, be aware of reality.


'Sex with my son is incredible - we're in love and we want a baby'

Ben Ford, who ditched his wife when he met his mother Kim West after 30 years, claims what the couple are doing 'isn't incest'

MIRROR.CO.UK

Former Shelbyville ISD teacher who had sex with underage student gets 3 years in prison

After a two day break over the weekend, A Shelby County jury was back in the courtroom looking to conclude the trial of a former Shelbyville ISD teacher who had...

KLTV.COM | BY CALEB BEAMES

Woman sent to jail for the rest of her life after raping her four grandchildren is described as the 'most evil person' the judge has ever seen

Edwina Louis rape...

See More

Woman sent to jail for raping her four grandchildren

A Ohio grandmother has been sentenced to four consecutive life terms after being found guilty of the rape of her own grandchildren. Edwina Louis, 53, will spend the rest of her life behind bars.

DAILYMAIL.CO.UK

http://www.thenativecanadian.com/.../eastern-ontario-teacher-...


The N.C. Chronicles.: Eastern Ontario teacher charged with 36 sexual offences

anti feminism, Child abuse, children's rights, Feminist hypocrisy,

THENATIVECANADIAN.COM | BY BLACKWOLF

Hyd woman kills newborn boy as she wanted daughter - Times of India

Having failed to bear a daughter for the third time, a shopkeeper's wife slit the throat of her 24day-old son with a shaving blade and left him to die in a street on Tuesday night.Purnima's first child was a stillborn boy, followed by another boy born five years ago.

TIMESOFINDIA.INDIATIMES.COM

Montgomery's son, Alan Vonn Webb, took the stand and was a key witness in her conviction.

"I want to see her placed somewhere she can never do that to children ...

See More

Woman sentenced to 40 years in prison for raping her children

A Murfreesboro mother found guilty of raping her own children learned her fate on Wednesday.

WAFF.COM | BY DENNIS FERRIER

gentler sex? Violence against men.'s photo.


Women, the gentler sex? Violence against men.

Like Page

April 8 at 1:38am

In fact, the past decade has seen a dramatic increase in the number of incidents of women raping and sexually assaulting boys and men. On May 2014, Jezebel repo...

End violence against women . . . .


North Carolina Grandma Eats Her Daughter's New Born Baby After Smoking Bath Salts

Henderson, North Carolina– A North Carolina grandmother of 4 and recovering drug addict, is now in custody after she allegedly ate her daughter's newborn baby....

AZ-365.TOP

28-Year-Old Texas Teacher Accused of Sending Nude Picture to 14-Year-Old Former Student

BREITBART.COM

http://latest.com/.../attractive-girl-gang-lured-men-alleywa.../

Attractive Girl Gang Lured Men Into Alleyways Where Female Body Builder Would Attack Them

A Mexican street gang made up entirely of women has been accused of using their feminine wiles to lure men into alleyways and then beating them up and...

LATEST.COM

http://www.wfmj.com/.../youngstown-woman-convicted-of-raping-...

Youngstown woman convicted of raping a 1 year old is back in jail

A Youngstown woman who went to prison for raping a 1-year-old boy fifteen years ago is in trouble with the law again.

WFMJ.COM

End violence against women . . . .

Women are raping boys and young men

Rape advocacy has been maligned and twisted into a political agenda controlled by radicalized activists. Tim Patten takes a razor keen and well supported look into the manufactured rape culture and...

AVOICEFORMEN.COM | BY TIM PATTEN

Bronx Woman Convicted of Poisoning and Drowning Her Children

Lisette Bamenga researched methods on the Internet before she killed her son and daughter in 2012.

NYTIMES.COM | BY MARC SANTORA

A Russian-born newlywed slowly butchered her German husband — feeding strips of his flesh to their dog until he took his last breath. Svetlana Batukova, 46, was...

See More

She killed her husband and then fed him to her dog: police

A Russian-born newlywed butchered her German hubby — and fed strips of his flesh to her pooch, authorities said. Svetlana Batukova offed Horst Hans Henkels at their...

NYPOST.COM

Daily Mail
January 15, 2015 ·

Mother charged with rape and sodomy of her son's 12-year-old friend

Mom, 30, 'raped and had oral sex with her son's 12-year-old friend'

Nicole Marie Smith, 30, (pictured) of St Charles County, Missouri, has been jailed after she allegedly targeted the 12-year-old boy at her home.

DAILYM.AI

April 4 at 4:48am ·

Female prison officers commit 90pc of sex assaults on male teens in US juvenile detention centres

Lawsuit in Idaho highlights the prevalence of sexual victimization of juvenile offenders.

IBTIMES.CO.UK | BY NICOLE ROJAS

This mother filmed herself raping her own son and then sold it to a man for $300. The courts just decide her fate. When you see what she got, you're going to be outraged.

Mother Who Filmed Herself Raping Her 1-Year-Old Son Receives Shocking Sentence

"...then used the money to buy herself a laptop..."

AMERICANEWS.COM

In several countries or rather in several regions of the world, family system has collapsed, due to bad nature and naughty acts of women. Particularly in Britain, and America, almost 50% people are alone, lonely, separated, divorced or failed marriages. In 2013, 48% children were born out of wedlock. It was projected that by 2016, more than 51% children will be born, to unmarried mothers. In these developed countries " paternity fraud " by women, are close to 20%. You can see several articles in the net, and in wikipedia etc. This means 1 out of 5 children are calling a wrong man as dad. The lonely, alone “ mothers " are frustrated. They see the children as burden. Love in the Society in general is lost, long time ago. The types of “ Mothers “ and “ Women “ we have now …………



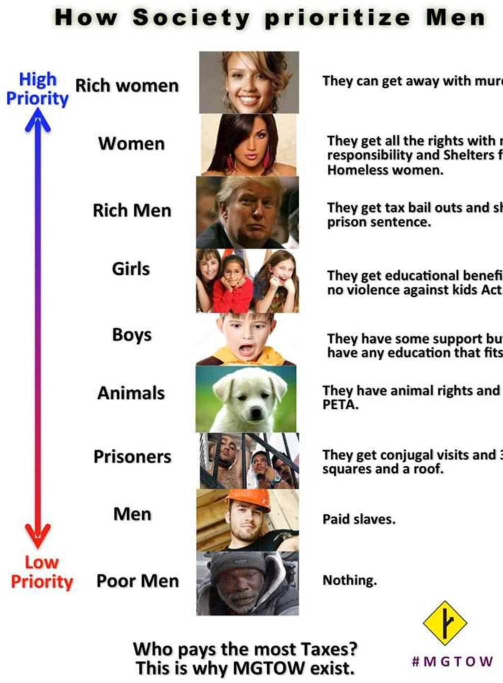




By now if you have assumed that Indian women are not doing any crime then please become friends with MRA Guri https://www.facebook.com/profile.php?id=100004138754180

He has dedicated his life to expose Indian Criminals





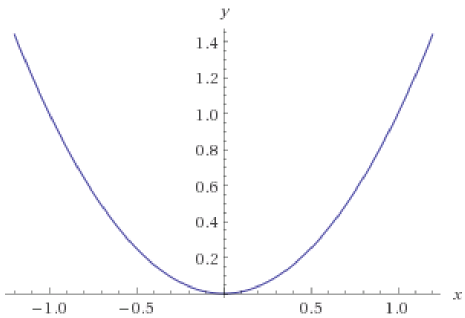

Professor Subhashish Chattopadhyay

**Spoon Feeding Series - Parabola**

Let us review the graphs of the Parbolas first

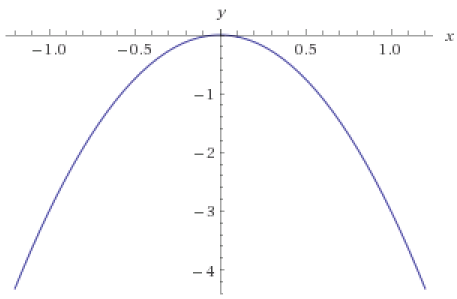
Graph of y = $x^2$ will be

plot $y = x^2$

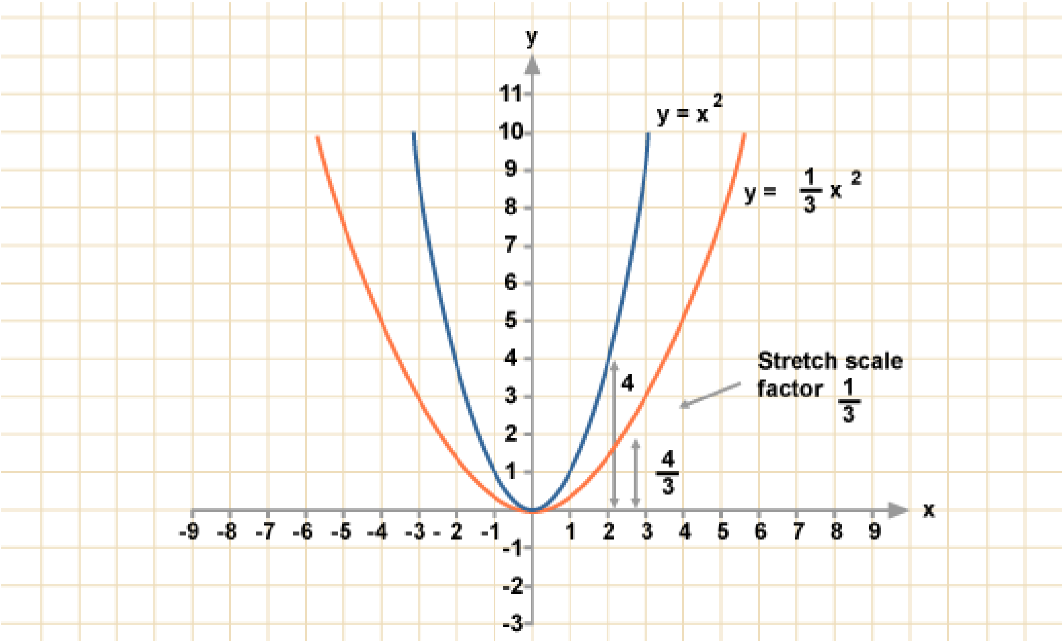



In contrast graph of y = $-3x^2$ will be downwards

plot $y = -3\,x^2$

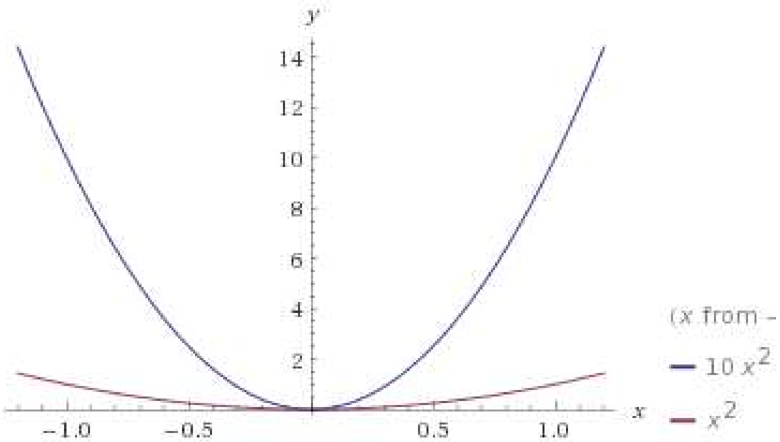



Graph of y = (1/3) $x^2$ will be flatter compared to y = $x^2$

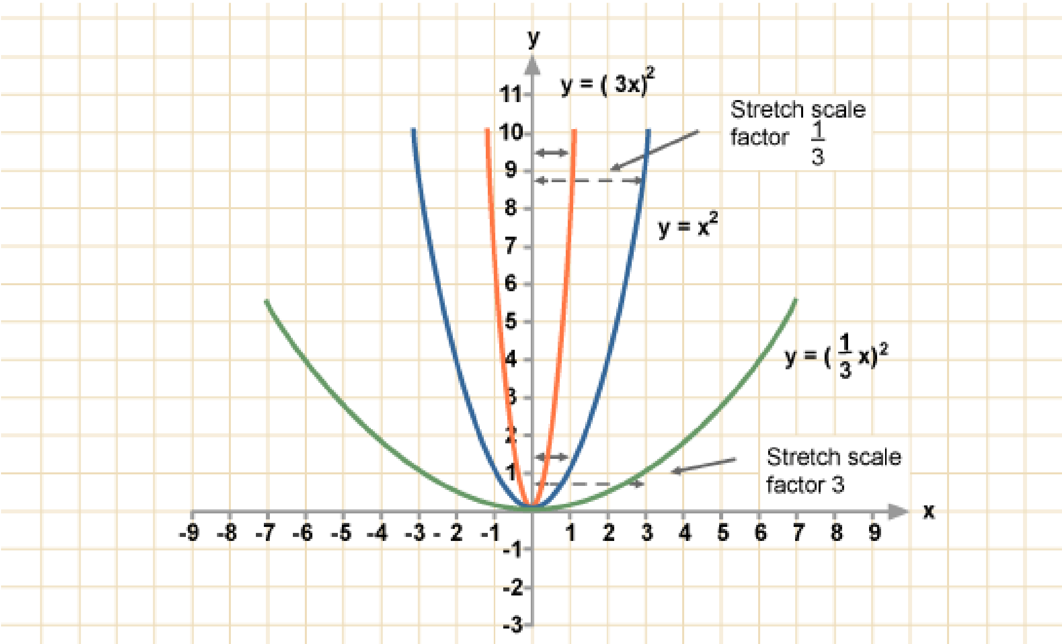



Similarly graph of $y = 10x^2$ will be narrow and steeper compared to $y = x^2$

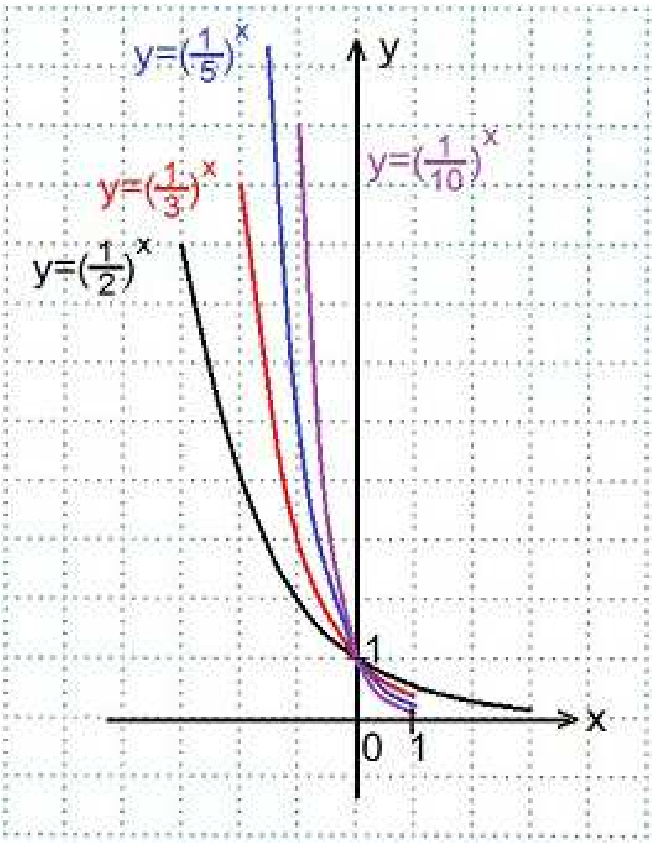
| plot | $y = 10\,x^2$ |
| --- | --- |
| | $y = x^2$ |



So see comparisons in a single image

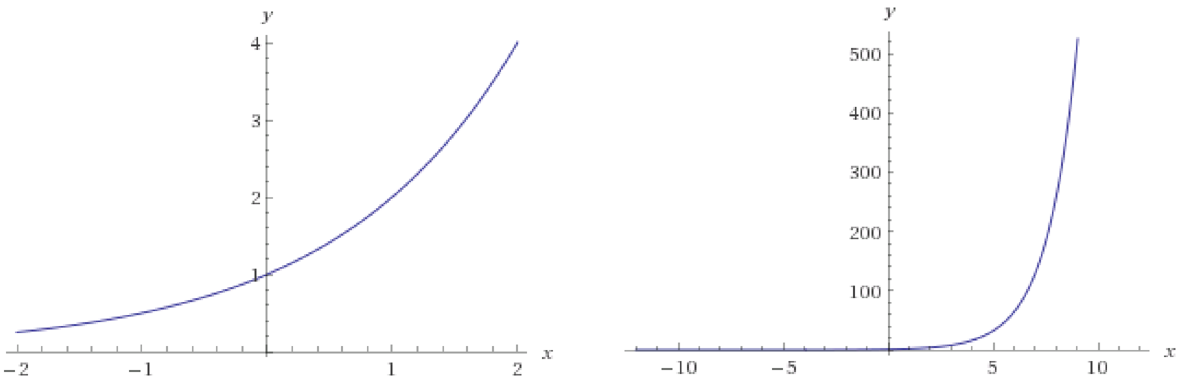



Similar things happen with power functions as well. Below we see fraction raised to power x



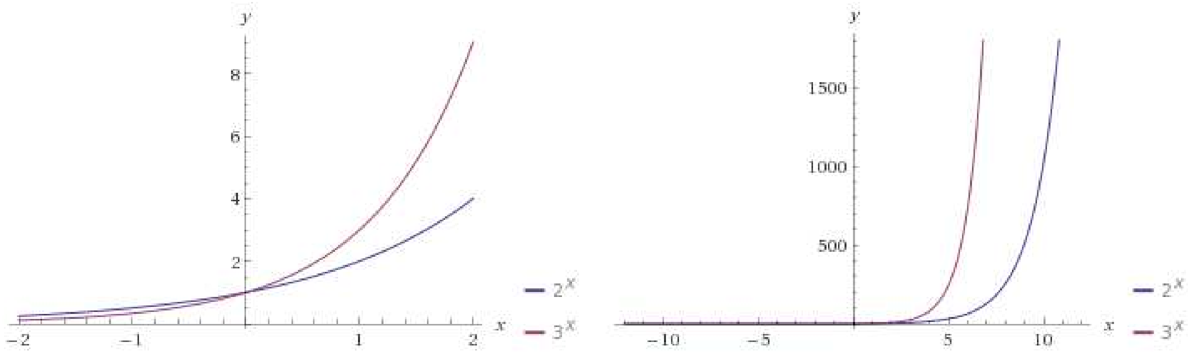
Let us see the graph of y = $2^x$

| plot | $y = 2^x$ |
|---|---|

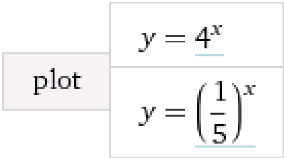


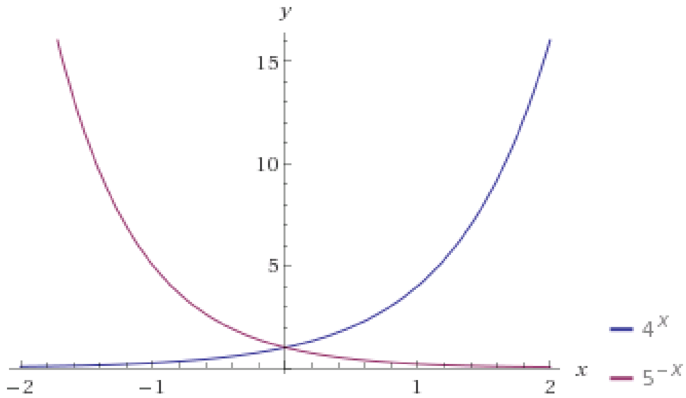

The graph of y = $3^x$ will be steeper and is understood easily by comparison

| plot | $y = 2^x$ |
|---|---|
| | $y = 3^x$ |

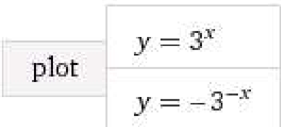


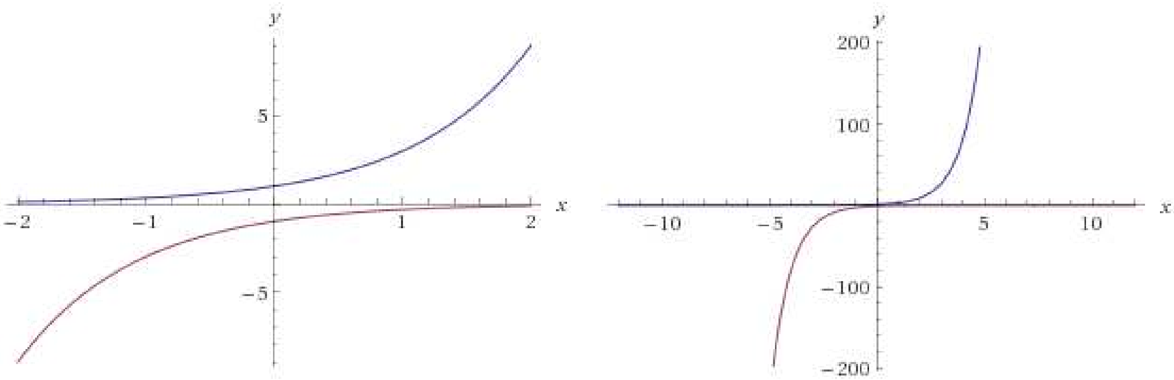
Now let us compare Integer to the power x and fraction to the power x



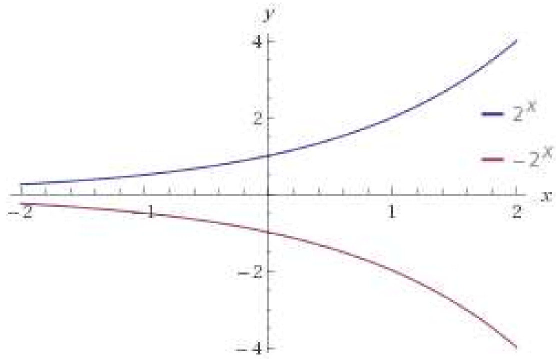

What about comparing y = $3^x$ and y = $-3^{-x}$

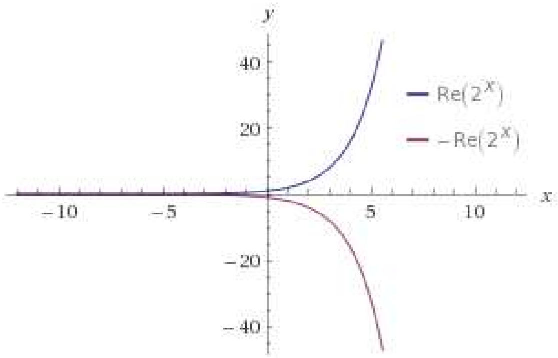


Spoon Feeding comparison of y = $2^x$ and y = $-2^x$

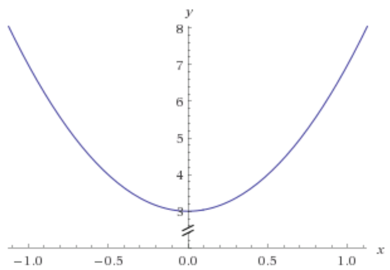
| plot | $y = 2^x$ |
|---|---|
| | $y = -2^x$ |

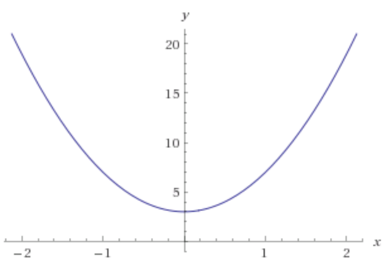



Graph of y = $4x^2 + 3$ will be 3 units above x-axis. So will pass through ( 0, 3 ) The parabola will look similar to y = $x^2$

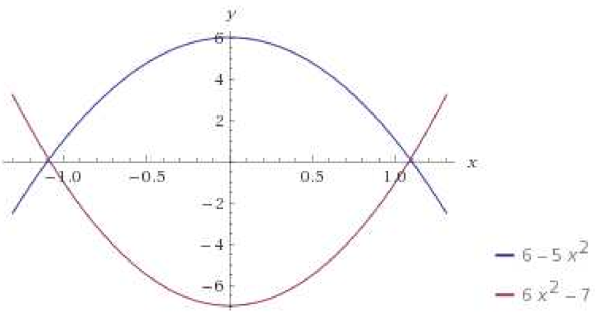
| plot | $y = 4x^2 + 3$ |
|---|---|

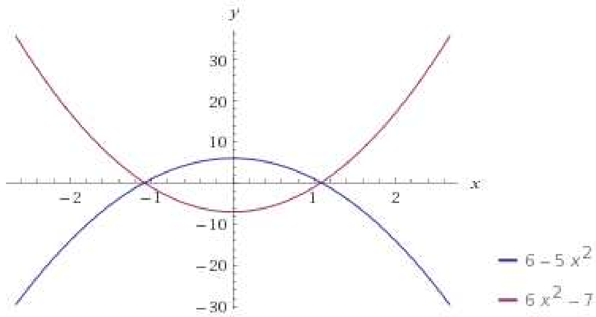


Let us learn more with graphs of y = $-5x^2 + 6$ and y = $6x^2 - 7$

plot $y = -5x^2 + 6$ ; $y = 6x^2 - 7$




Don't quickly assume that the graphs are intersecting on x axis. The roots are very close.

$5x^2 = 6 \Rightarrow x = \pm\sqrt{6/5} = \pm 1.095$

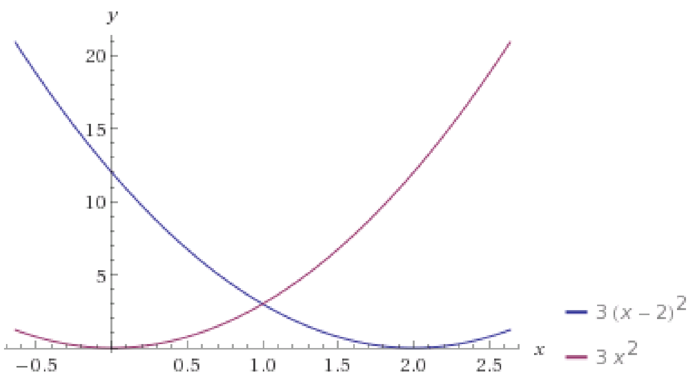
While $6x^2 = 7 \Rightarrow x = \pm\sqrt{7/6} = \pm 1.0801$

-

Concept of Shifting of graphs

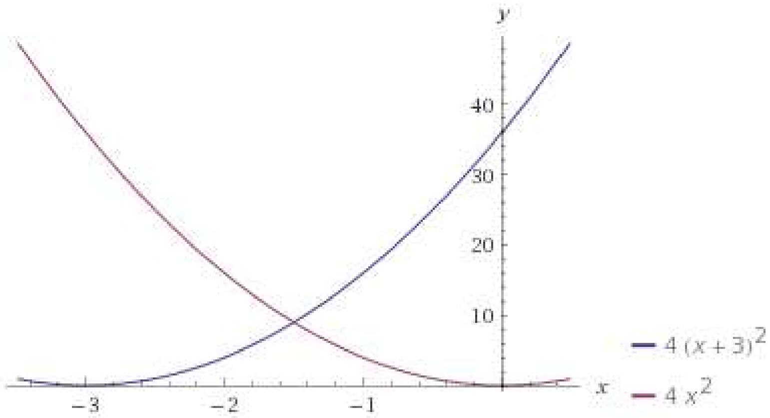
The graph of y = $3(x - 2)^2$ will be same as y = $3x^2$ while shifted by 2 units towards right

plot $y = 3(x-2)^2$ ; $y = 3x^2$



Similarly graph of y = 4( x + 3 )$^2$ will be shifted by 3 units on left compared to y = 4x$^2$ which is through the origin

plot $y = 4\,(x+3)^2$
$y = 4\,x^2$

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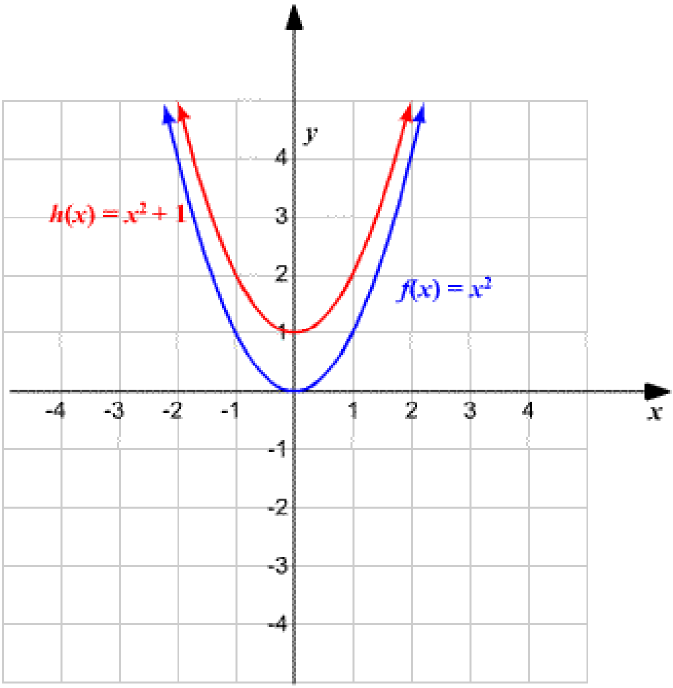
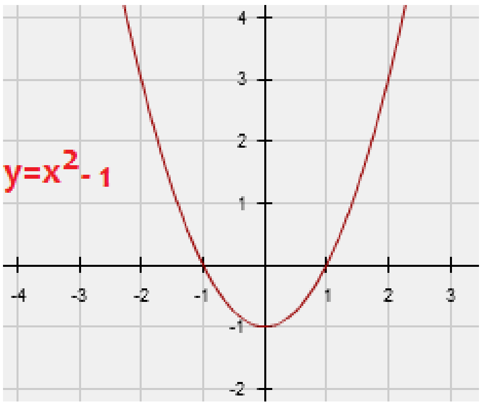
IIT-JEE 2005 Shifting a Parabola and then finding the area is discussed / explained at

https://archive.org/details/AreaDefiniteIntegralIITJEE2005ShiftingParabolasLeftOrRight




In the above image see how the upper graph is shifted up by 1 due to +1

In the image below the graph is shifted down by -1

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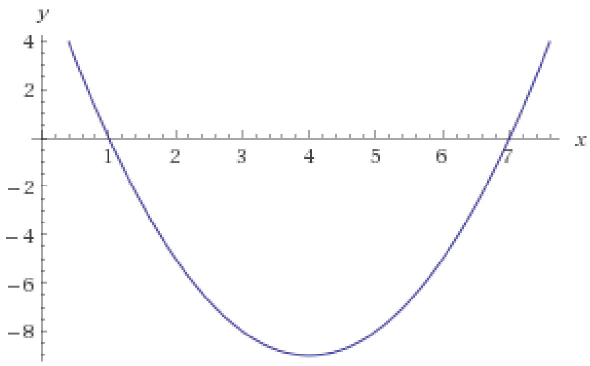
The parabola that passes through ( 1,0 ) and ( 7,0 ) will be ( x - 1 )( x - 7 )

In simple words the Quadratic expression that has roots 1 and 7 is a parabola through 1 and 7

So graph of y = (x - 1 )( x - 7 ) = $x^2$ - 8x + 7 is

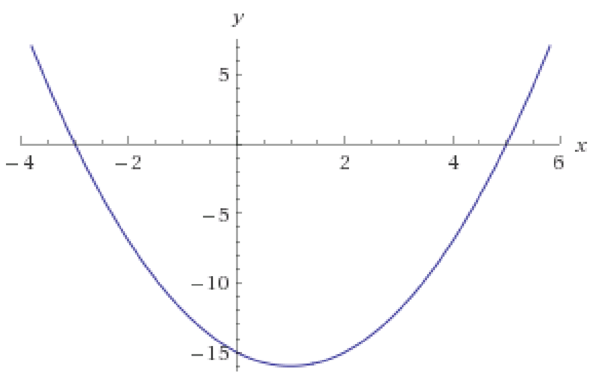
| plot | $y = x^2 - 8\,x + 7$ |
|---|---|




If a Quadratic expression has roots -3, 5 then it will be a parabola passing through -3 and 5

So graph of y = ( x + 3 ) ( x - 5 ) = $x^2$ -2x - 15 is

| plot | $y = x^2 - 2\,x - 15$ |
|---|---|

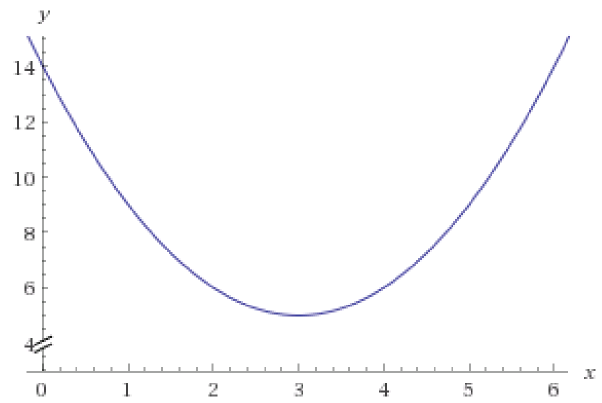



If the Discriminant D < 0 i.e. $b^2$ < 4ac then the whole parabola is above x-axis signifying imaginary roots. As the parabola does not intersect the x-axis at all. For a > 0

If a is negative then the parabola will be downwards

So graph of y = ( x - 3 )$^2$ + 5 will be

| plot | $y = (x-3)^2 + 5$ |
|---|---|

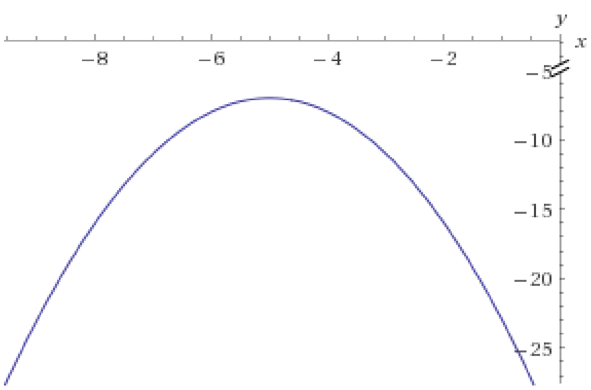



Meaning minima will be at x = 3 so $x^2$ graph shifted right by 3 and added 5 so moved up by 5 units

So we can easily guess the graph of y = - ( x + 5 )$^2$ -7 …..

It will be shifted left by 5 units. So maxima will be at x = -5 and 7 units below x axis

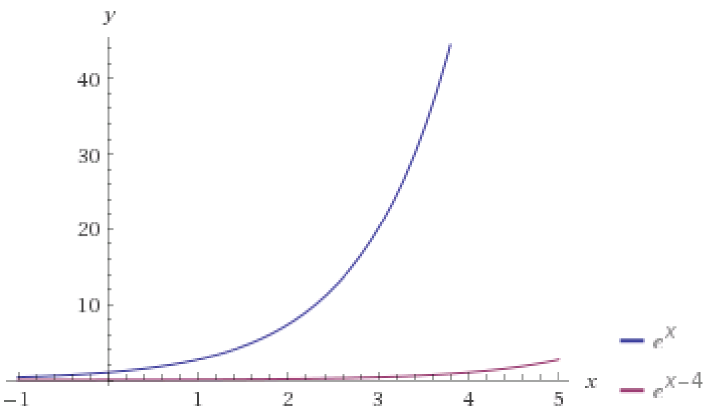
| plot | $y = -(x+5)^2 - 7$ |
|---|---|




The parabola is downwards because coeff of $x^2$ is -ve

Don't use the idea of shift blindly ! The graph of y = $e^{x-4}$ is not shifted by 4 units that of y = $e^x$

plot $y = e^x$, $y = e^{x-4}$

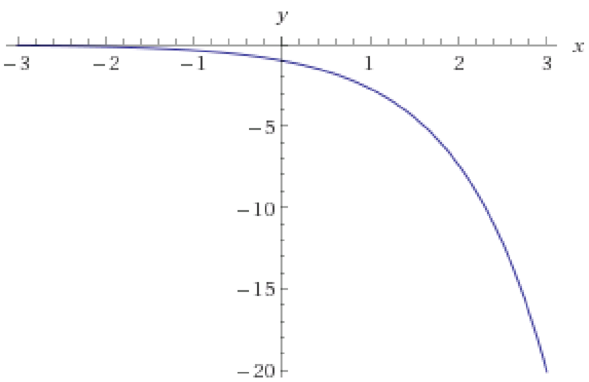



This is because $e^{(x-4)} = e^x / e^4$ means just divided by a value

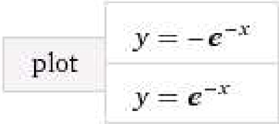
Concept of Reflections

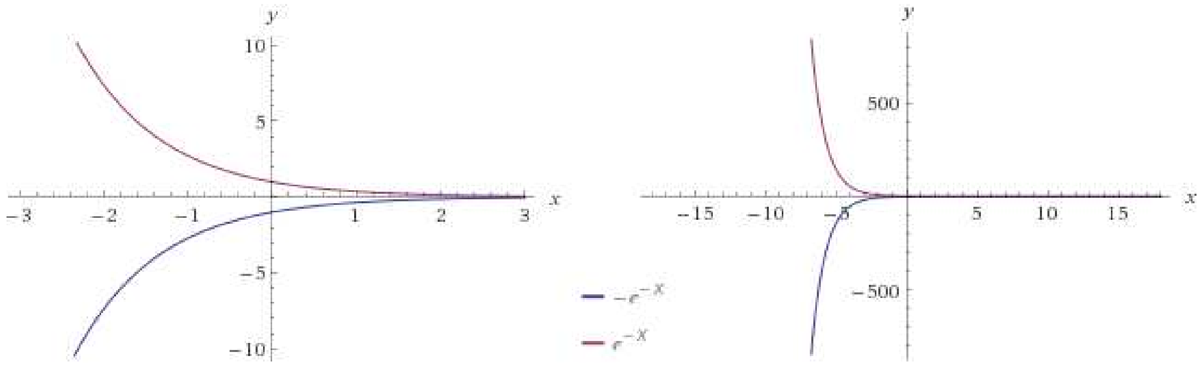
Guess the graph of y = - $e^x$

plot $y = -e^x$




What about graph of y = $e^{(-x)}$ and y = - $e^{(-x)}$

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IIT JEE 1984, 1992 Problems and Solutions as being discussed in the class. Explains various kinds of graphs at https://archive.org/details/AreaDefiniteIntegralIITJEE19841992TypesOfGraphs

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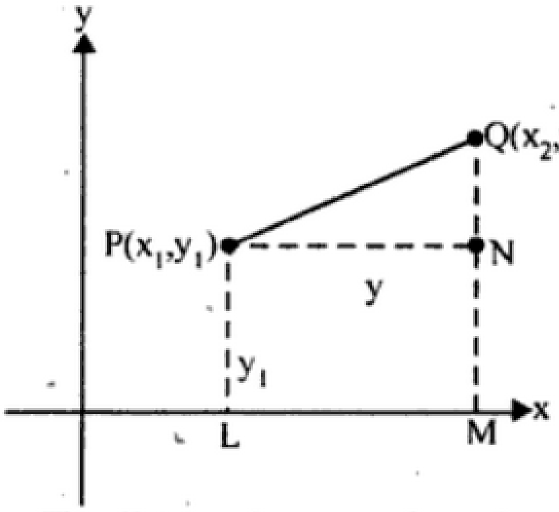
Before we discuss examples and problems let us see the formulae

**Distance between two points**

Here $QN = QM - NM = y_2 - y_1$

$PN = OM - OL = x_2 - x_1$




∴ The distance between the points $P(x_1, y_1)$ and $Q(x_2\ y_2)$ is given by

$$PQ^2 = PN^2 + QN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

i.e., $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Question

The point (4, 1) undergoes the following successive transformations:

(i) reflection about the line $y = x$

(ii) translation through a distance 2 units along the positive $x$-axis.

then, the final coordinates of the point are

(a) (4, 3) (b) (3, 4) (c) (1, 4) (d) (4, 4)

*Ans.* (b)

**Solution** Let $Q(x, y)$ be the reflection of $P(4, 1)$ about the line $y = x$, then mid-point of $PQ$ lies on this line and $PQ$ is perpendicular to it. So we have

$\frac{y+1}{2} = \frac{x+4}{2}$ and $\frac{y-1}{x-4} = -1.$

$\Rightarrow \quad x - y = -3$ and $x + y = 5$

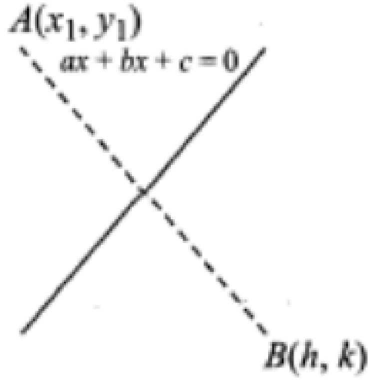
$\Rightarrow \quad x = 1, y = 4$

Therefore reflection of (4, 1) about $y = x$ is (1, 4). Next, this point is shifted. 2 units along the positive $x$-axis, the new coordinates are $(1 + 2, 4 + 0) = (3, 4)$

Image of a point

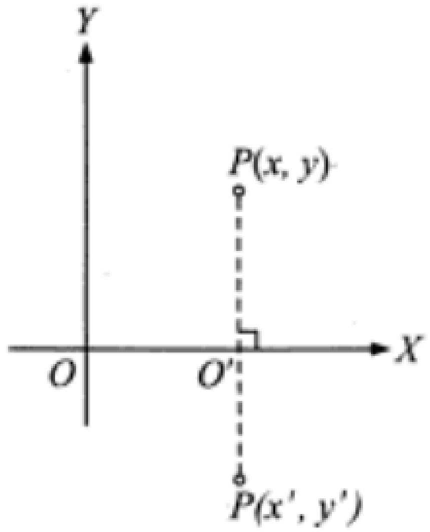
**The image of a point with respect to the line mirror.** The image of $A(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(h, k)$ given by,

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$



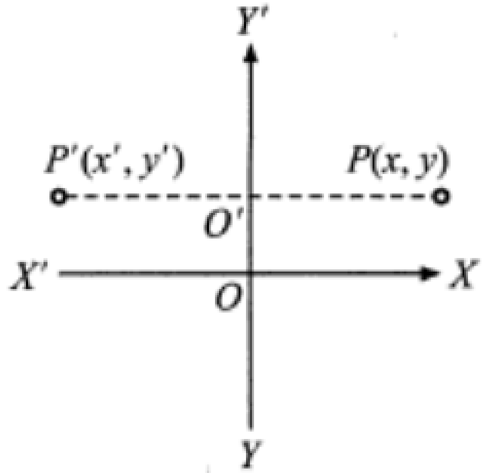

**The image of a point with respect to *x*-axis:** Let $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the $x$-axis, then

$x' = x$ and $y' = -y$, ($\because$ $O'$ is the mid point of $PP'$)



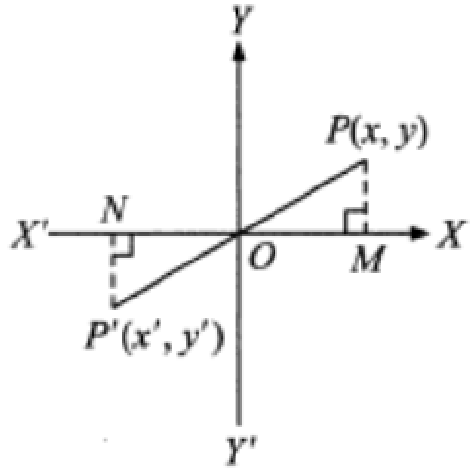
**The image of a point with respect to y-axis:** $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the $y$-axis, then

$x' = -x$ and $y' = y$ ($\because$ $O'$ is the mid point of $PP'$)



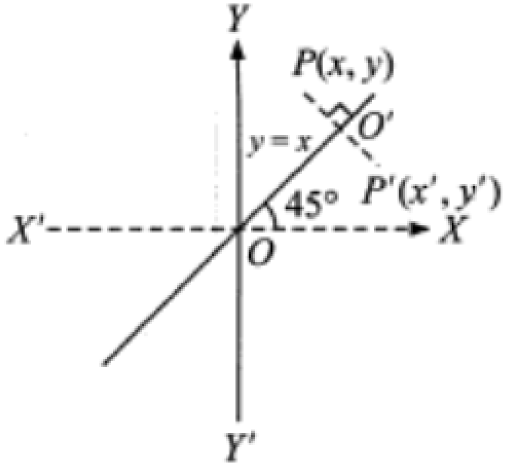

**The image of a point with respect to the origin:** Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection through the origin, then

$x' = -x$ and $y' = -y$ ($\because$ $O$ is the mid-point of $PP'$)

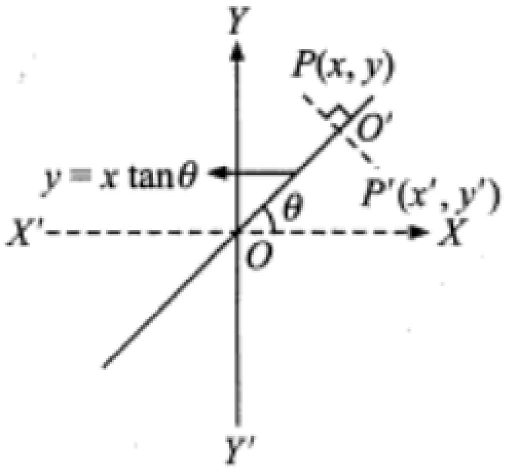


**The image of a point with respect to the line $y = x$:** Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection in the line $y = x$, then,

$x' = y$ and $y' = x$ $\quad$ ($\because$ $O'$ is the mid-point of $PP'$)




**The image of a point with respect to the line $y = x \tan\theta$:** Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection in the line $y = x \tan \theta$, then,

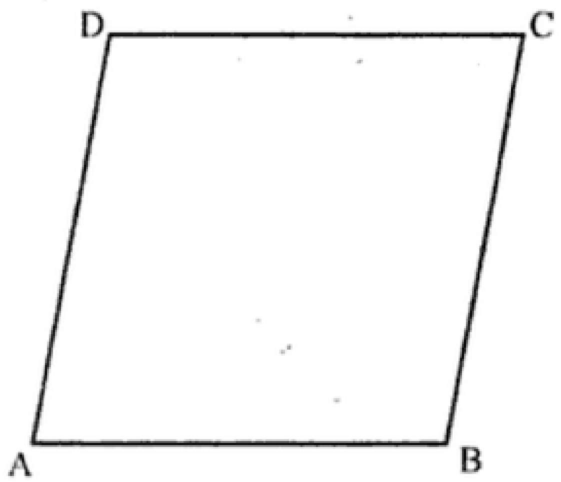



$x' = x \cos 2\theta + y \sin 2\theta$

$y' = x \sin 2\theta - y \cos 2\theta,$

($\because$ $O'$ is the mid-point of $PP'$)

A Rhombus is made by distorting a square




All four sides are equal. So AB = BC = CD = DA

Question

The diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0$, $mx + ly + n = 0$, $mx + ly + n' = 0$ include an angle

(a) $\frac{\pi}{3}$

(b) $\frac{\pi}{2}$

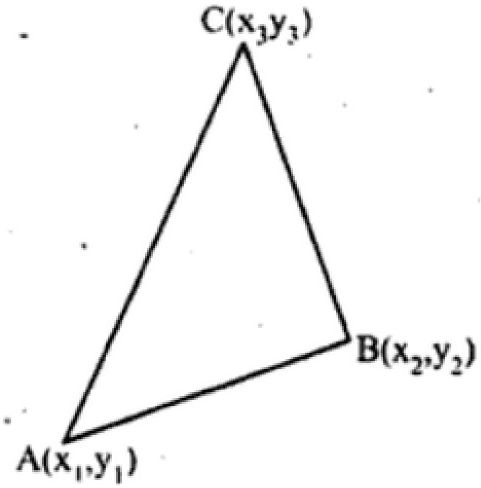
(c) $\tan^{-1}\left(\frac{l^2 - m^2}{l^2 + m^2}\right)$

(d) $\tan^{-1}\left(\frac{2lm}{l^2 + m^2}\right)$

Solution

**(b).** Since the distance between the parallel lines $lx + my + n = 0$ and $lx + my + n' = 0$ is same as the distance between the parallel lines $mx + ly + n = 0$ and $mx + ly + n' = 0$. Therefore, the parallelogram is a rhombus. Since the diagonals of a rhombus are at right angles, therefore the required angle is $\frac{\pi}{2}$.

Area of a Triangle




**The area of a triangle, the coordinates of whose vertices are $(x_1, y_1)$ $(x_2, y_2)$ and $(x_3, y_3)$ is given by**

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

or

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Question on Area

Let $P\ (2, -4)$ and $Q\ (3, 1)$ be two given points. Let $R\ (x, y)$ be a point such that $(x - 2)(x - 3) + (y - 1)(y + 4) = 0$. If area of $\Delta PQR$ is $\frac{13}{2}$, then the number of possible positions of $R$ are

(a) 2 (b) 3

(c) 4 (d) none of these

Solution

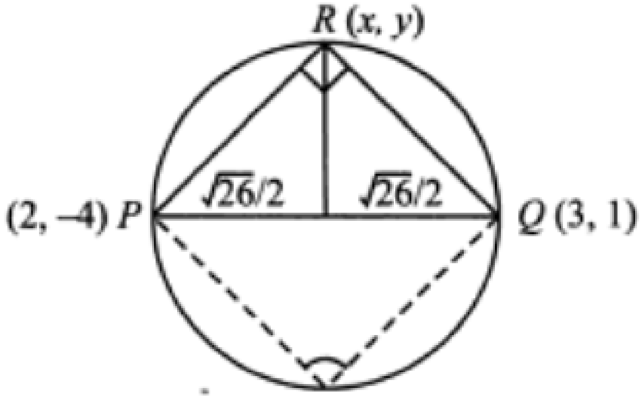
**(a).** We have

$$(x-2)(x-3)+(y-1)(y+4)=0$$

$$\Rightarrow \quad \left(\frac{y+4}{x-2}\right)\times\left(\frac{y-1}{x-3}\right)=-1$$

$$\Rightarrow \quad RP \perp RQ \text{ or } \angle PRQ = \frac{\pi}{2}.$$

$\therefore$ The point $R$ lies on the circle whose diameter is $PQ$.




Now, area of $\Delta PQR = \frac{13}{2}$

$$\Rightarrow \quad \frac{1}{2}\times\sqrt{26}\times(\text{altitude})=\frac{13}{2}$$

$$\Rightarrow \quad \text{altitude} = \frac{\sqrt{26}}{2} = \text{radius}$$

$\Rightarrow$ there are **two** possible positions of $R$.

Condition of colinearity of 3 points

**Three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear if**

**i) Area of triangle ABC = 0 i.e.,**

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

**or**

**ii) AB+BC=AC (or) AC+BC=AB (or) AC+AB=BC**

In some cases a problem can be solved just by observation. Meaning the above determinant need not be evaluated.

The points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ are

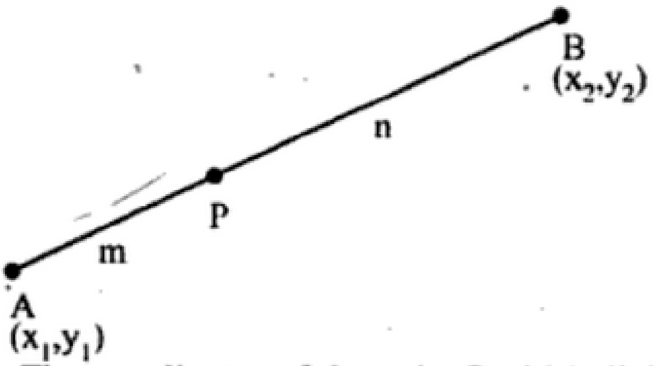
(a) vertices of an equilateral triangle
(b) concyclic
(c) vertices of a right angled triangle
(d) none of these

*Ans.* (d)

**Solution** As the given points lie on the line $x + y = a + b + c$, they are collinear.

-

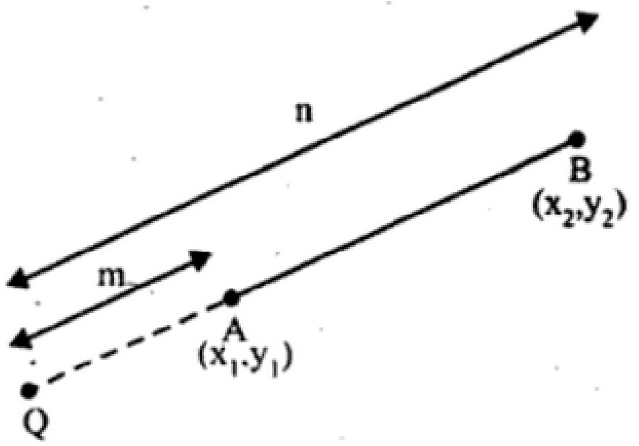
Section formula Internal Division




**The coordinates of the point P which divides the line segment joining the points $A(x_1\ y_1)$ and $B(x_2\ y_2)$ internally in the ratio m:n are given by**

$$P=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$

Section formula External Division can have Two formulae. Depending on from which external side the division is being done




Here the external point Q is on the side of A

If m is the distance from A then m gets multiplied to coordinates of opposite point i.e.

B( $x_2$ , $y_2$ )

The coordinates of the point Q which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2\ y_2)$ externally in the ratio m:n are given by

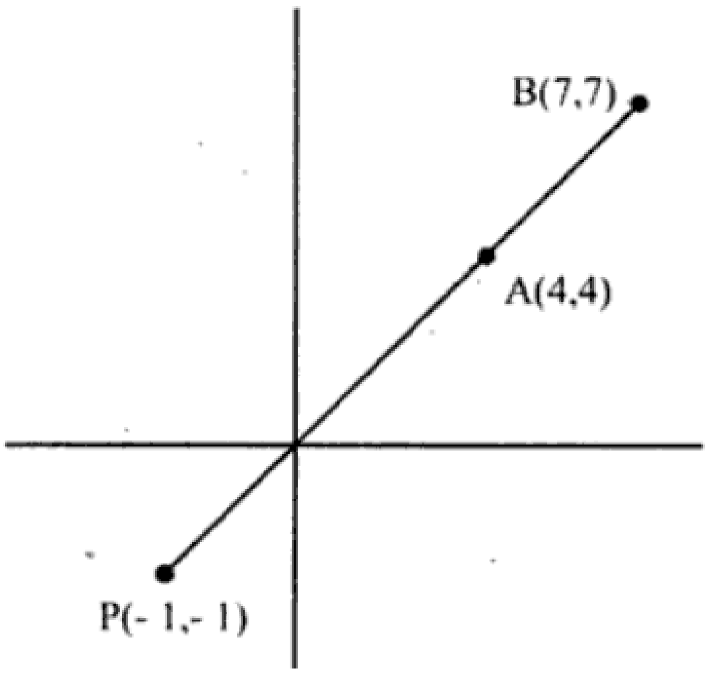
$$Q = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

**Note:**

**i)** If P is the mid point of AB, then the coordinate of P is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

**ii)** The co-ordinate of any point on AB can be written as $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

**The ratio in which the line joining the points A (4, 4) and B(7, 7) is divided by (-1, -1)**



**a)** 7:4 externally **b)** 8:5 externally
**c)** 5:8 externally **d)** 4:7 externally

**Ans (c)**

$$PA = \sqrt{5^2 + 5^2} = 5\sqrt{2}$$

$$PB = \sqrt{8^2 + 8^2} = 8\sqrt{2}$$

$$\therefore \quad PA : PB = 5 : 8$$

thus P (-1, -1) divides AB externally in the ratio 5:8.

Question

If two vertices of a triangle are (–2, 3) and (5, –1), orthocentre lies at the origin and centroid on the line $x + y = 7$, then the third vertex lies at

(a) (7, 4) (b) (8, 14)
(c) (12, 21) (d) none of these

*Ans.* (d)

**Solution** Let $O(0, 0)$ be the orthocentre; $A(h, k)$ the third vertex; and $B(-2, 3)$ and $C(5, -1)$ the other two vertices. Then the slope of the line through $A$ and $O$ is $k/h$, while the line through $B$ and $C$ has the slope $(-1-3)/(5+2) = -4/7$. By the property of the orthocentre, these two lines must be perpendicular, so we have

$$\left(\frac{k}{h}\right)\left(-\frac{4}{7}\right) = -1 \Rightarrow \frac{k}{h} = \frac{7}{4} \qquad \text{(i)}$$

Also
$$\frac{5-2+h}{3} + \frac{-1+3+k}{3} = 7$$

$\Rightarrow$
$$h + k = 16 \qquad \text{(ii)}$$

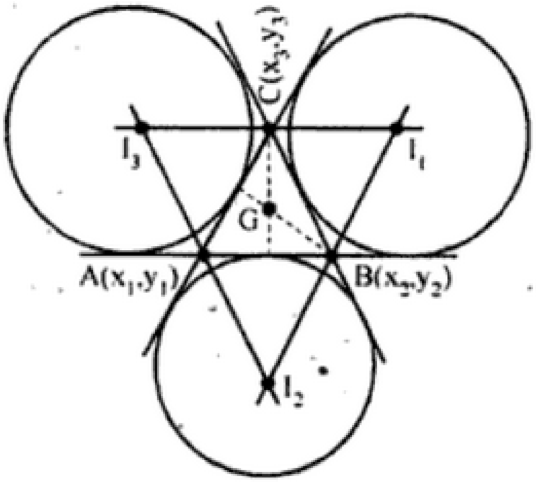
Which is not satisfied by the points given in (a), (b), or (c)

-

## Coordinates of the centroid, in-centre and ex-centres of a triangle

Let $A(x_1,y_1)$ $B(x_2\ y_2)$ and $C(x_3\ y_3)$ be the three vertices of a triangle ABC.

### i) Centroid of a triangle




Centroid is the point of intersection of medians, whose coordinates are given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

### ii) In-centre of a triangle

In-centre is the point of intersection of internal angular bisectors, whose coordinates are given by

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where a, b, c are the lengths of the sides BC, CA, AB respectively.

**iii) Ex-centres of a triangle**

The point of intersection $I_1$ of the external angular bisectors of $\angle B$ and $\angle C$ is one of the excentres of the triangle ABC and is given by

$$I_1 = \left( \frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c} \right)$$

similarly the other ex-centres are given by

$$I_2 = \left( \frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c} \right) \text{ and}$$

$$I_3 = \left( \frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c} \right)$$

where a, b, c are the lengths of the sides BC, CA, AB respectively.

Question

If the vertices $P$, $Q$, $R$ of a $\Delta PQR$ are rational points, which of the following points of the $\Delta PQR$ is (are) always rational point(s)?

(a) centroid (b) incentre

(c) circumcentre (d) orthocentre

(A rational point is a point both of whose coordinates are rational numbers)

Solution

**(a).** Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$; $R \equiv (x_3, y_3)$, where $x_i$, $y_i$ $(i = 1, 2, 3)$ are rational numbers.

Now, the centroid of $\Delta PQR$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

which is rational point. Incentre, circumcentre and orthocentre depend on sides of the triangle which may not be rational even if vertices are so. For example, for $P$ (0, 1) and $Q$ (1, 0); $PQ = \sqrt{2}$.

Question

**Let A (-1, 5) B (3,1) C(5, a) be the vertices of a triangle ABC. If D, E, F are the middle points of BC, CA and AB respectively and area of triangle ABC is equal to four times the area of triangle DEF, then**

**a)** a = 3 **b)** a ≠ 5

**c)** for any real value of a

**d)** any real value except -1.

**Ans (d)**

Since A, B, C from a triangle

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \neq 0 \text{ i.e., } \begin{vmatrix} 1 & -1 & 5 \\ 1 & 3 & 1 \\ 1 & 5 & a \end{vmatrix} \neq 0$$

$4a + 4 \neq 0 \Rightarrow a \neq -1$

**but since area of any triangle is always four times the area of a triangle formed by the mid points a, can be any real value except -1.**

In some problems we find the Area pretty differently

The area of the triangle formed by the tangent to the curve $y = \dfrac{8}{4+x^2}$ at $x = 2$ and the coordinate axes is

(*a*) 2 sq. units (*b*) $\frac{7}{2}$ sq. units

(*c*) 4 sq. units (*d*) 8 sq. units.

Solution

**(c) From** $y = \dfrac{8}{4+x^2}$,

**when** $x = 2, y = \dfrac{8}{4+4} = 1$

**Also,** $\dfrac{dy}{dx} = -\dfrac{8}{(4+x^2)^2}(2x) \Rightarrow \left[\dfrac{dy}{dx}\right]_{(2,\,1)} = -\dfrac{1}{2}$

$\therefore$ **equation of tangent is**

$$y - 1 = -\frac{1}{2}(x-2) \quad \text{or} \quad x + 2y - 4 = 0 \qquad ...(1)$$

**Its intercepts on axes are (by putting** $y = 0$ **and** $x = 0$ **respectively)** $a = 4, b = 2$

$\therefore$ **Area** $= \dfrac{1}{2}ab = \dfrac{1}{2} \times 4 \times 2 = 4$ **sq. units.**

-

Perpendicular Lines

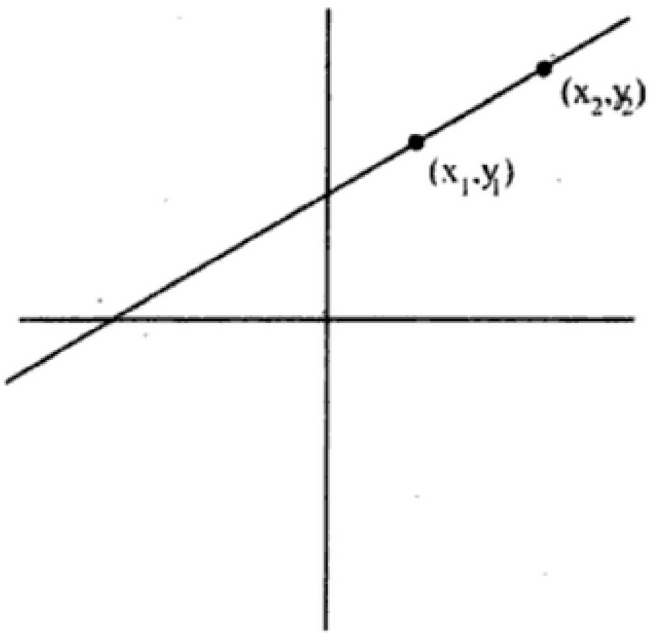
If there is a line whose slope is m ( assuming this line NOT parallel to x-axis ) then the slope of the line which is perpendicular to this will be -1 / m

Meaning, product of the slopes of lines that are perpendicular is -1

If one of the lines is parallel to x-axis its slope is 0 while the line perpendicular will have a slope of infinity ( ∞ ) This line is parallel to y-axis. Product of 0 X ∞ is undefined. In this case we do not apply the -1 as product rule.

-

Equation of the line passing through two points




The equation of a line passing through two points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

-

The intercept form of a line

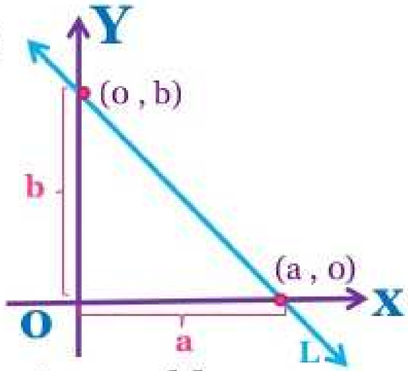
- Suppose a line **L** makes x-intercept a and y-intercept b on the axes. Obviously **L** meets x-axis at the point (a, 0) and y-axis at the point (0, b).

By two-point form of the equation of the line, we have

$$y - 0 = \frac{b-0}{0-a}(x-a)$$

Or $ay = -bx + ab$

i.e., $\frac{x}{a} + \frac{y}{b} = 1$




Thus, equation of the line making intercepts **a** and **b** on x- and y-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Question

Through the point $P(\alpha, \beta)$, where $\alpha\beta > 0$ the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area $S$. If $ab > 0$, then the least value of $S$ is

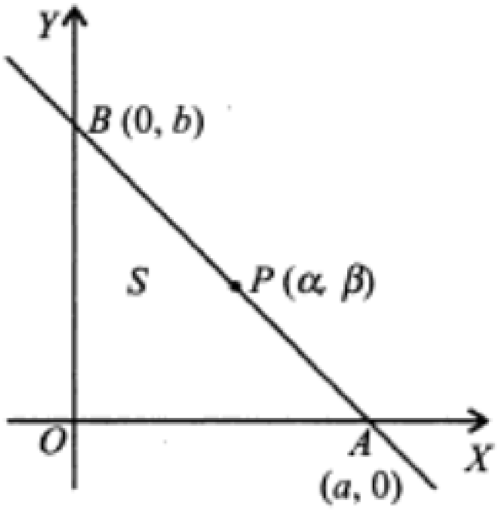
(a) $\alpha\beta$ (b) $2\alpha\beta$

(c) $4\alpha\beta$ (d) none of these

Solution

**(b).** The equation of the given line is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ...(1)$$




This line cuts $x$-axis and $y$-axis at $A$ $(a, 0)$ and $B$ $(0, b)$ respectively.

Since area of $\Delta OAB = S$ (Given)

$\therefore \quad \left|\frac{1}{2}ab\right| = S$ or $ab = 2S \quad (\because\ ab > 0) \qquad ...(2)$

Since the line (1) passes through the point $P$ $(\alpha, \beta)$

$\therefore \quad \frac{\alpha}{a}+\frac{\beta}{b}=1$ or $\frac{\alpha}{a}+\frac{a\beta}{2S}=1 \qquad$ [Using (2)]

or $\quad a^2\beta - 2aS + 2\alpha S = 0.$

Since $a$ is real, $\therefore\ 4S^2 - 8\alpha\beta S \geq 0$

or $\quad 4S^2 \geq 8\alpha\beta S$ or $S \geq 2\alpha\beta \quad \left(\because\ S=\frac{1}{2}ab>0 \text{ as } ab>0\right)$

Hence the least value of $S = 2\alpha\beta$.

-

i) The equation of a line parallel to a given line $ax+by+c=0$ is $ax+by+\lambda=0$, where $\lambda$ is constant.

ii) The equation of a line perpendicular to a given line $ax+by+c=0$ is $bx-ay+\lambda=0$, where $\lambda$ is constant.

iii) The slope of the line $ax+by+c=0$ is given by

$$m = \frac{-a}{b}.$$

iv) For intercept on x-axis, put $y=0$. For intercept on y-axis, put $x=0$.

v) Angle $\theta$ between the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ is given by

$$\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

vi) The lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ are

a) Coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

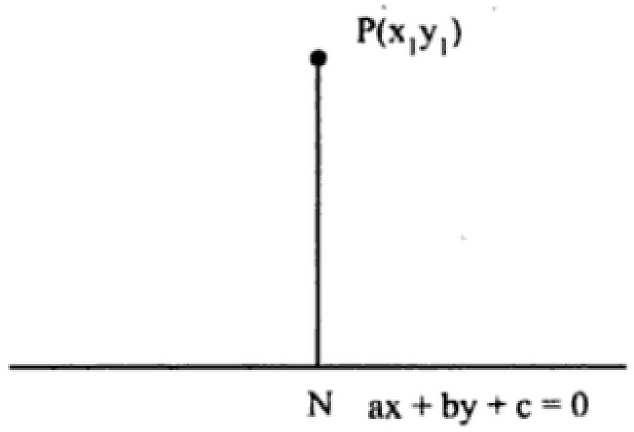
b) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

c) intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

d) Perpendicular if $a_1a_2 + b_1b_2=0$

Distance of a point from a line

**The length of the perpendicular from a point $(x_1, y_1)$ to a line $ax+by+c=0$ is given by**




$$PN = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

**Note:**

**The length of the perpendicular from the origin to the line $ax+by+c=0$ is** $\dfrac{|c|}{\sqrt{a^2+b^2}}$

Question on Length of Perpendiculars

If $p_1, p_2$ denote the lengths of the perpendiculars from the point (2, 3) on the lines given by $15x^2 + 31xy + 14y^2 = 0$, then if $p_1 > p_2$, $p_1^2 + \frac{1}{74} - p_2^2 + \frac{1}{13}$ is equal to

(a) $-2$ (b) 0
(c) 2 (d) none of these

*Ans.* (c)

**Solution** The lines given by $15x^2 + 31xy + 14y^2 = 0$ are

$$5x + 7y = 0 \text{ and } 3x + 2y = 0$$

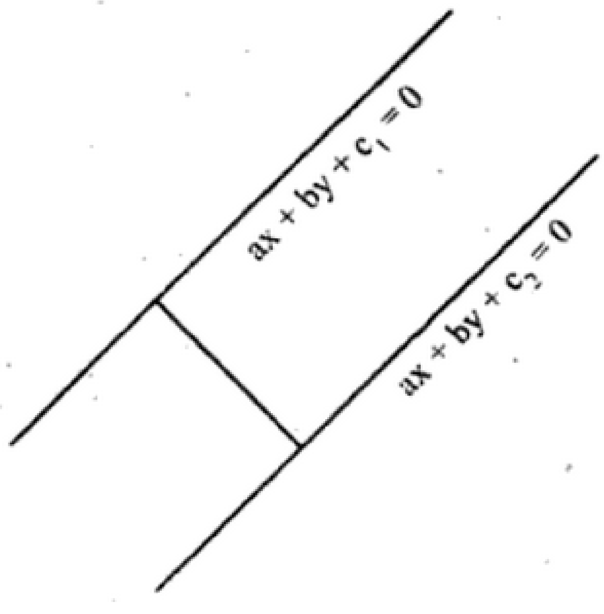
Length of the perpendiculars from (2, 3) on these lines are

$$p_1 = \frac{31}{\sqrt{74}} \text{ and } p_2 = \frac{12}{\sqrt{13}} \qquad [\because p_1 > p_2]$$

So that $p_1^2 + \frac{1}{74} - p_2^2 + \frac{1}{13} = \frac{961}{74} + \frac{1}{74} - \left(\frac{144}{13} - \frac{1}{13}\right) = 2.$

**The distance between the parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is given by**

$$\frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}.$$



The two points $(x_1, y_1)$ and $(x_2, y_2)$ are on the same (or opposite) sides of the straight line $ax+by+c=0$ according to the quantities $ax_1+by_1+c$ and $ax_2+by_2+c$ have same (or opposite) signs.

Question

The distance between the parallel lines given by $(x+7y)^2 + 4\sqrt{2}\,(x+7y) - 42 = 0$ is

(a) 4/5 (b) $4\sqrt{2}$ (c) 2 (d) $10\sqrt{2}$

*Ans.* (c)

**Solution** The lines given by the equation are

$$(x + 7y - 3\sqrt{2})(x + 7y + 7\sqrt{2}) = 0$$

$$\Rightarrow \quad x + 7y - 3\sqrt{2} = 0 \quad \text{and} \quad x + 7y + 7\sqrt{2} = 0$$

$$\text{distance between these lines} = \left|\frac{7\sqrt{2} - (-3\sqrt{2})}{\sqrt{1^2 + 7^2}}\right| = 2.$$

The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are *concurrent* (intersect at a point) if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Question

Lines $ax + by + c = 0$ where $3a + 2b + 4c = 0$, $a, b, c \in \mathbf{R}$ are concurrent at the point.

(a) (3, 2) (b) (2, 4) (c) (3, 4) (d) (3/4, 1/2)

*Ans.* (d)

**Solution** $3a + 2b + 4c = 0$

$$\Rightarrow \quad \frac{3}{4}a + \frac{1}{2}b + c = 0$$

$\Rightarrow ax + by + c$ passes through (3/4, 1/2) for all values of $a, b, c$.

Question

If the lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4cy + c = 0$ are concurrent, then $a, b, c$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

*Ans.* (c)

**Solution** Since the given lines are concurrent

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0 \quad \Rightarrow \quad -bc + 2ac - ab = 0$$

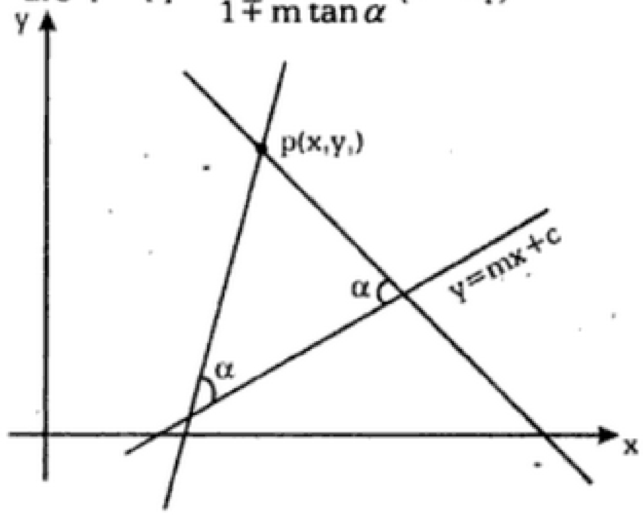
$$\Rightarrow \quad b = \frac{2ac}{a+c}$$

$\Rightarrow$ $a, b, c$ are in H.P.

-

**The equations of the straight lines which pass through a given point $(x_1, y_1)$ and make a given angle $\alpha$ with the given straight line $y=mx+c$**

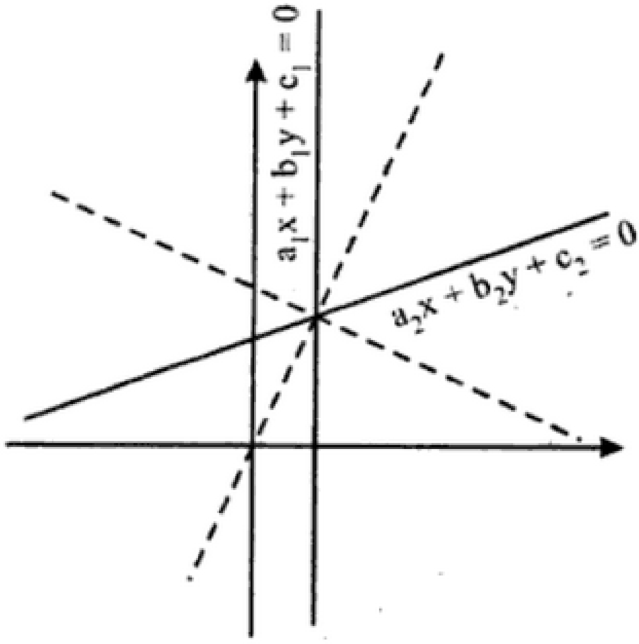
are $y - y_1 = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$




**The angle between the lines $x\cos\alpha_1 + y\sin\alpha_1 = P_1$ and $x\cos\alpha_2 + y\sin\alpha_2 = P_2$ is $\alpha_1 - \alpha_2$.**

Equation of Internal and External bisectors of 2 Lines

The equation of the bisectors of the angles between the lines $a_1x+b_1y+c_1 = 0$ and $a_2x+b_2y+c_2=0$ is given by




$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

**Bisector of the angle containing the origin**

If $c_1$, $c_2$ are positive, then the equation of the bisector of the angle containing the origin is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Bisector of Acute and Obtuse angle between lines

i) If $c_1$, $c_2$ are positive and if $a_1a_2+b_1b_2>0$, then

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$ is the obtuse angle bisector and

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$ is the acute angle bisector.

ii) If $c_1$,$c_2$ are positive and if $a_1a_2+b_1b_2<0$, then

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

is the acute angle bisector and

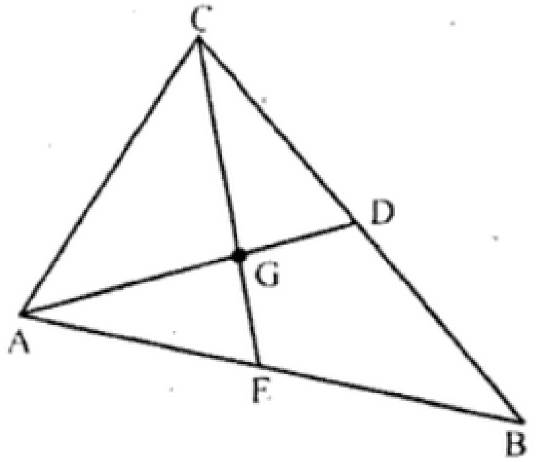
$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

is the obtuse angle bisector.

If $c_1$, $c_2$ are positive and $a_1a_2+b_1b_2>0$, then the origin lies in the obtuse angle and the '+' sign gives the bisector of the obtuse angle. If $a_1a_2+b_1b_2<0$, then the origin lies in the acute angle and '+' sign gives the bisector of acute angle.

Coordinates of Centroid, Orthocenter, Circumcenter of a Triangle

**Centroid:** The point of intersection of the medians of a triangle is called its centroid. It divides the median in the ratio 2:1.




If $(x_1, y_1)$ $(x_2, y_2)$ and $(x_3, y_3)$ are the vertices of a triangle, then the coordinates of its centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Question on Centroid

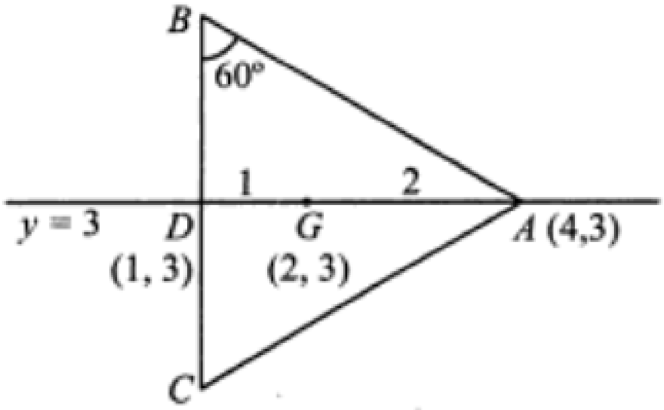
If the centroid and a vertex of an equilateral triangle are (2, 3) and (4, 3) respectively, then the other two vertices of the triangle are

(a) $(1, 3 \pm \sqrt{3})$ (b) $(2, 3 \pm \sqrt{3})$

(c) $(1, 2 \pm \sqrt{3})$ (d) $(2, 2 \pm \sqrt{3})$

Solution

**(a).** $G$ being the centroid, divides $AD$ in the ratio 2 : 1.




Since $AG = 2$, $\quad \therefore \quad GD = 1$,

$\therefore$ Coordinates of $D$, using section formula, are $D(1, 3)$.

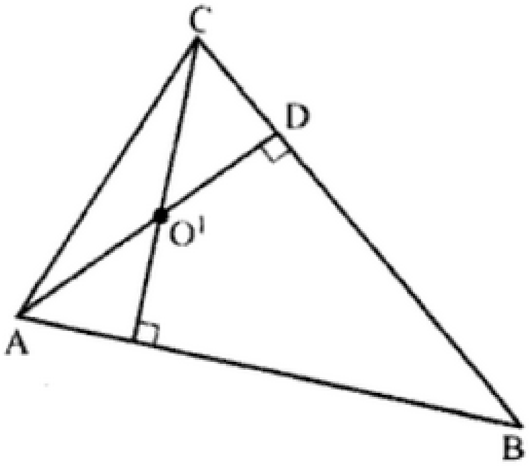
Now $AD = 1 + 2 = 3$, $\therefore$ $\tan 60° = \dfrac{3}{BD} \Rightarrow BD = \sqrt{3}$.

$\therefore \quad B \equiv (1, 3 + \sqrt{3})$ and $C \equiv (1, 3 - \sqrt{3})$.

## Orthocentre

The point of intersection of the altitudes of a triangle is called its orthocentre

To determine the orthocentre, first we find equations of line passing through vertices and perpendicular to the opposite sides. Solving two of these three equations we get the co-ordinates of orthocentre.




If angles $A$, $B$ and $C$ and vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of a $\Delta ABC$ are given, then orthocentre of $\Delta ABC$ is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

If any two lines out of three lines, i.e., $AB$, $BC$ and $CA$ are perpendicular, then orthocentre is the point of intersection of two perpendicular lines.

The orthocentre of the triangle with vertices (0, 0), $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\left\{(y_1 - y_2)\left[\frac{x_1 x_2 - y_1 y_2}{x_2 y_1 - x_1 y_2}\right],\right.$$

$$\left.(x_1 - x_2)\left[\frac{x_1 x_2 + y_1 y_2}{x_1 y_2 - x_2 y_1}\right]\right\}$$

Question on Orthocenter

The orthocentre of the triangle formed by the lines $xy = 0$ and $2x + 3y - 5 = 0$ is

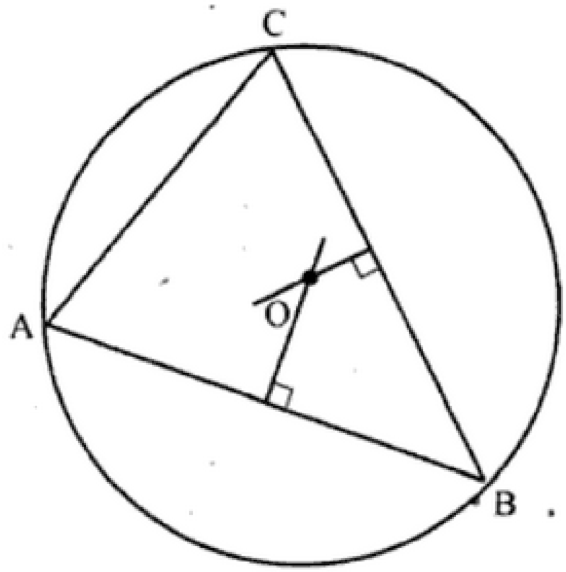
(a) (2, 3) (b) (3, 2) (c) (0, 0) (d) (5, – 5)

*Ans.* (c)

**Solution** The given triangle is right angled at (0, 0) which is therefore the orthocentre of the triangle.

## Circumcentre

The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circum-centre. It is equidistant from the vertices of a triangle.




**Note:**

The circumcentre O, centroid G and orthocentre O′ of a triangle ABC are collinear such that G divides O′O in the ratio 2:1
i.e., O′G:OG=2:1

Question

If the circumcentre of a triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$; then the orthocentre lies on the line

(a) $y = (a^2 + 1)x$ (b) $y = 2ax$

(c) $x + y = 0$ (d) $(a-1)^2 x - (a+1)^2 y = 0$

*Ans.* (d)

**Solution** We know from geometry that the circumcentre, centroid and orthocentre of a triangle lie on a line. So the orthocentre of the triangle lies on the line joining the circumcentre (0, 0) and the centroid $\left(\frac{(a+1)^2}{2}, \frac{(a-1)^2}{2}\right)$

i.e. $$\frac{(a+1)^2}{2} y = \frac{(a-1)^2}{2} x$$

or $$(a-1)^2 x - (a+1)^2 y = 0.$$

Question

If the equations of the sides of a triangle are $x + y = 2$, $y = x$ and $\sqrt{3}y + x = 0$, then which of the following is an exterior point of the triangle?

(a) orthocentre (b) incentre

(c) centroid (d) none of these

Solution

**(a).** The lines $y = x$ and $\sqrt{3}y + x = 0$ are inclined at $45^\circ$ and $150^\circ$, respectively, with the positive direction of $x$-axis. So, the angle between the two lines is an obtuse angle. Therefore, orthocentre lies outside the given triangle, whereas incentre and centroid lie within the triangle (In any triangle, the centroid and the incentre lie within the triangle).

Question

The equations to the sides of a triangle are $x - 3y = 0$, $4x + 3y = 5$ and $3x + y = 0$. The line $3x - 4y = 0$ passes through the

(a) incentre (b) centroid

(c) circumcentre (d) orthocentre of the triangle

*Ans.* (d)

**Solution** Two sides $x - 3y = 0$ and $3x + y = 0$ of the triangle being perpendicular to each other, the triangle is right angled at the origin, the point of intersection of these sides. So that origin is the orthocentre of the triangle and the line $3x - 4y = 0$ passes through this orthocentre.

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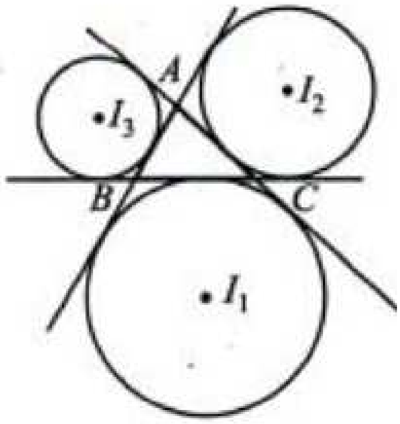



**Ex-Centres of a Triangle** A circle touches one side outside the triangle and the other two extended sides then circle is known as excircle.

Let $ABC$ be a triangle then there are three excircles, with three excentres $I_1$, $I_2$, $I_3$ opposite to vertices $A$, $B$ and $C$ respectively. If the vertices of triangle are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ then

$$I_1 = \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

$$I_2 = \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right).$$

-

**Family of lines through the intersection of two given lines**

The equation of family of lines passing through the intersection of the lines

$L_1 = a_1x+b_1y+c_1=0$ and $L_2=a_2x+b_2y+c_2=0$ is

$(a_1x+b_1y+c_1) + \lambda (a_2x+b_2y+c_2) = 0$, where $\lambda$ is a parameter i.e., $L_1+\lambda L_2=0$.

Formulae specific to Pair of Straight Lines

**Homogeneous equation of second degree in x and y**

A general homogenous equation of degree 2 always represent two straight lines, real or imaginary, through the origin. Conversely, the equal of a pair of lines through origin is a second degree homogeneous equation in x and y.

The equation of the form $ax^2+2hxy+by^2=0$ is called a homogeneous equation of degree 2 in x and y, where a, b, h are constants.

let $ax^2 + 2hxy+by^2=0$ ...(1)

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of Straight lines only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{i.e., iff} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

For easy remembering note that the first row of the Determinant is coeffs of x terms

$(a)x^2 + 2(h)xy$ …. + 2 (g) x …..

Similarly the second row is made of coeffs of y terms. i.e.

2 ( h ) xy + $(b)y^2$ + 2 (f) y ….

The last row of the determinant is the last 3 constants of last 3 terms. i.e. g, f, and c

*Equation of the lines joining the origin to the points of intersection of a line and a conic.*

Let $L \equiv lx + my + n = 0$

and $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

be the equations of a line and a *conic*, respectively. Writing the equation of the line as $\dfrac{lx+my}{-n} = 1$ and making $S = 0$ homogeneous with its help, we get

$$S = ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

which being a homogeneous equation of second degree, represents a pair of straight lines through the origin and passing through the points common to $S = 0$ and $L = 0$.

-

Equation of the pair of lines through the origin perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Question

If the slope of one of the lines represented by $ax^2 - 6xy + y^2 = 0$ is square of the other, then

(a) $a = 1$ (b) $a = 2$ (c) $a = 4$ (d) $a = 8$

*Ans.* (d)

**Solution** Let the lines represented by the given equation be $y = mx$ and $y = m^2x$, then

$$m + m^2 = 6 \text{ and } m^3 = a$$

$\Rightarrow$ $m = 2$ or $-3$

and so $a = 8$ or $-27$

Question

If the pairs of lines $x^2 + 2xy + ay^2 = 0$ and $ax^2 + 2xy + y^2 = 0$ have exactly one line in common then the joint equation of the other two lines is given by

(a) $3x^2 + 8xy - 3y^2 = 0$
(b) $3x^2 + 10xy + 3y^2 = 0$
(c) $y^2 + 2xy - 3x^2 = 0$
(d) $x^2 + 2xy - 3y^2 = 0$

*Ans.* (b)

**Solution** Let $y = mx$ be a line common to the given pairs of lines, then

$$am^2 + 2am + 1 = 0 \text{ and } m^2 + 2m + a = 0 \Rightarrow \frac{m^2}{2(1-a)} = \frac{m}{a^2-1} = \frac{1}{2(1-a)}$$

$$\Rightarrow m^2 = 1 \text{ and } m = -\frac{a+1}{2} \Rightarrow (a+1)^2 = 4 \Rightarrow a = 1 \text{ or } -3$$

But for $a = 1$, the two pairs have both the lines common, so $a = -3$ and the slope $m$ of the line common to both the pairs is 1.

Now $x^2 + 2xy + ay^2 = x^2 + 2xy - 3y^2 = (x - y)(x + 3y)$

and $ax^2 + 2xy + y^2 = -3x^2 + 2xy + y^2 = -(x - y)(3x + y)$

So the equation of the required lines is

$$(x + 3y)(3x + y) = 0 \Rightarrow 3x^2 + 10xy + 3y^2 = 0.$$

Question on Locus

If $P$ (1, 0), $Q$ (– 1, 0) and $R$ (2, 0) are three given points. The point $S$ satisfies the relation $SQ^2 + SR^2 = 2SP^2$. The locus of $S$ meets $PQ$ at the point

(a) (0, 0) (b) (2/3, 0)

(c) (– 3/2, 0) (d) (0, – 2/3)

*Ans.* (c)

**Solution** Let $S$ be the point $(x, y)$

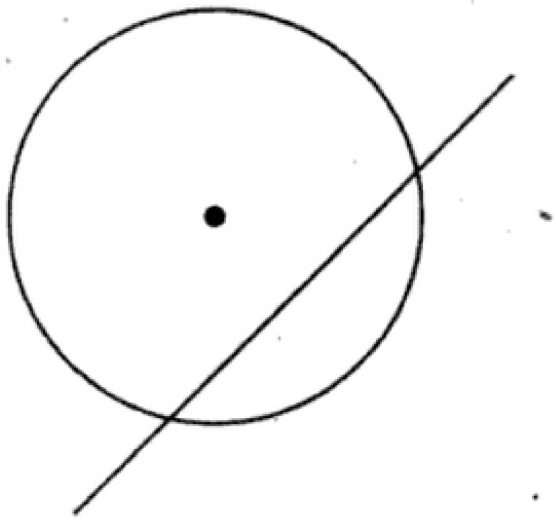
then $$(x + 1)^2 + y^2 + (x - 2)^2 + y^2 = 2[(x - 1)^2 + y^2]$$

$\Rightarrow 2x + 3 = 0$, the locus of $S$ and equation of $PQ$ is $y = 0$.

So the required points is (– 3/2, 0).

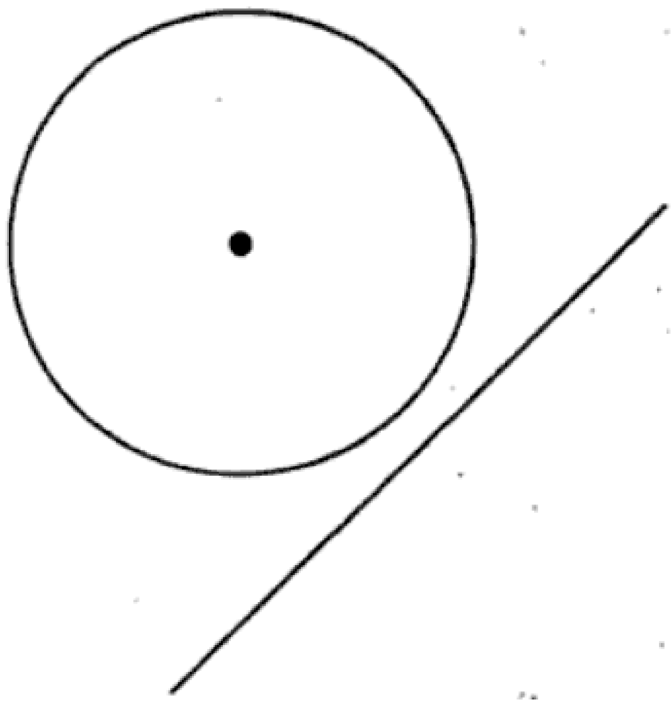
Formulae related to circles

The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ at two distinct points if the length of the perpendicular from the centre is less than the radius of the circle.



ie., $\left|\dfrac{c}{\sqrt{1+m^2}}\right| < a$

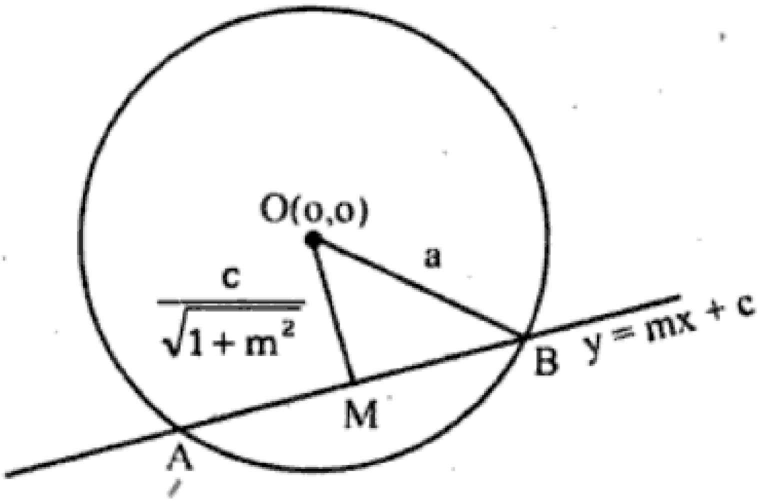
The line does not intersect the circle $x^2 + y^2 = a^2$ if the length of the perpendicular, from the centre is greater than the radius of the circle



ie., $\left|\dfrac{c}{\sqrt{1+m^2}}\right| > a$

iii) The length of the intercept cut off from a line $y = mx + c$ by a circle $x^2 + y^2 = a^2$ is

$$2MB = 2\sqrt{\frac{a^2(1+m^2)-c^2}{(1+m^2)}}$$




Question on Tangent

The point on the curve $y = 6x - x^2$ where the tangent is parallel to $x$-axis is

(a) (0, 0) (b) (2, 8)
(c) (6, 0) (d) (3, 9).

Solution

(d) $\frac{dy}{dx} = 6 - 2x$

$\therefore \frac{dy}{dx} = 0 \Rightarrow x = 3.$

$\therefore y = 18 - 9 = 9 \quad \therefore$ Point is (3, 9).

Question

For the curve $x = t^2 - 1, y = t^2 - t$, the tangent line is perpendicular to $x$- axis, where

(a) $t = 0$ (b) $t \to \infty$

(c) $t = \frac{1}{\sqrt{3}}$ (d) $t = -\frac{1}{\sqrt{3}}$.

Solution

(a) $\frac{dx}{dt} = 2t$,

Tangent is perpendicular to $x$-axis if $\frac{dx}{dt} = 0 \Rightarrow t = 0$.

Question

The point on the curve $y^2 = x$, the tangent at which makes an angle of 45° with $x$-axis will be given by

(a) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) $(2, 4)$ (d) $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Solution

(d) $y^2 = x \Rightarrow 2y\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y} = \tan 45° = 1$ (given)

$\Rightarrow y = \frac{1}{2}$. $\therefore x = \frac{1}{4}$

$\therefore$ Point is $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Question

**If tangent to the curve $x = at^2, y = 2at$ is perpendicular to $x$-axis then its point of contact is**

(a) $(a, a)$ (b) $(0, a)$

(c) $(a, 0)$ (d) $(0, 0)$.

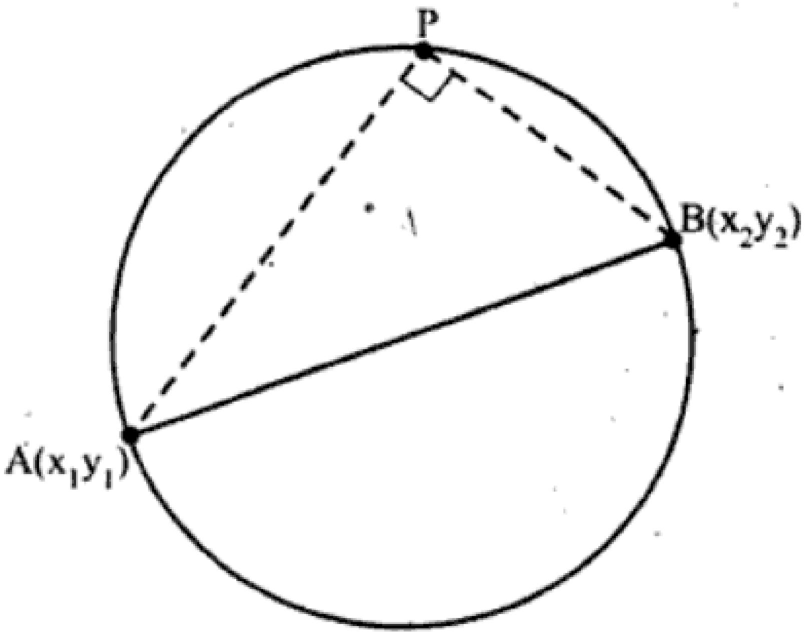
Solution

(d) $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a \Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow \frac{1}{t} = \infty \Rightarrow t = 0 \Rightarrow$ Point is $(0, 0)$.

Equation of the circle when the end points of a diameter are given

**Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of circle and let P be any point on circle.**



**Now, since the angle subtended at the point P in the semicircle APB is a right angle.**

**$m_1 m_2 = -1$ ( $m_1$ = slope of AP, $m_2$ = slope of BP)**

$$\frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

**ie., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$**

## Condition for two intersecting circles to be orthogonal

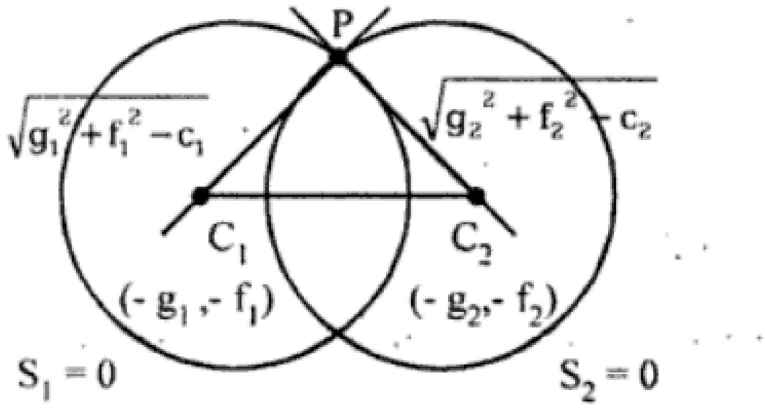
### Definition

Two intersecting circles are said to cut each other orthogonally when the tangents at the point of intersection of the two circles are at right angles.

Let the circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + C_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + C_2 = 0$




intersect orthogonally, then $\angle C_1PC_2 = 90°$

ie., $\Delta C_1PC_2$ is right angled

$\therefore C_1C_2^2 = C_1P^2 + C_2P^2$

$(g_1 - g_2)^2 + (f_1 - f_2)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2)$

$\Rightarrow 2g_1g_2 + 2f_1f_2 = c_1 + c_2$ is the required condition that $S_1$ and $S_2$ intersect orthogonally.

## Some important results

i) The equation of chord joining two points $\theta_1$ and $\theta_2$ on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$(x + g)\cos\frac{\theta_1+\theta_2}{2} + (y + f)\sin\frac{\theta_1+\theta_2}{2} = r\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$, where r is the radius of the circle.

ii) The equation of the tangent at $P(\theta)$ on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x+g)\cos\theta + (y + f)\sin\theta = \sqrt{g^2+f^2-c}$

iii) The locus of the point of intersection of two tangents drawn to the circle $x^2 + y^2 = a^2$ which makes an constant angle $\alpha$ to each other is $x^2 + y^2 - 2a^2 = 4a^2(x^2 + y^2 - a^2)\cot^2\alpha$.

Question

**The equation of tangent to the circle $x^2 + y^2 + 6x + 4y - 12 = 0$ at (6,2) is**

**a)** $4x - 9y - 6 = 0$ **b)** $9x + 4y + 12 = 0$

**b)** $3x - 9y = 0$ **d)** $2x - 3y = 6$

**Ans (b)**

**Note:**

The equation of tangent at $(x_1, y_1)$ is

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

thus the equation of tangent at (6,2) is

$6x + 2y + 3(x+6) + 2(y+2) - 12 = 0$

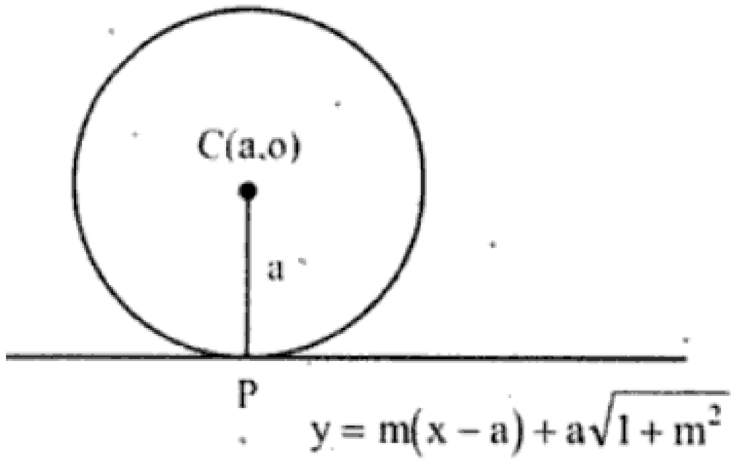
i.e., $9x + 4y + 12 = 0$.

Question

**The line** $y = m(x-a)+a\sqrt{1+m^2}$ **touches the circle** $x^2 + y^2 = 2ax$

a) for only two real values of m
b) for only one real value of m
c) for no real value of m
d) for all real values of m

**Ans (d)**




The centre and radius of the circle $x^2 + y^2 - 2ax$ are (a, 0) and a respectively.
The length of perpendicular from (a, 0) to the line $y - mx + am - a\sqrt{1+m^2} = 0$ is

$$CP = \left|\frac{0 - ma + am - a\sqrt{1+m^2}}{\sqrt{1+m^2}}\right| = a$$

since the $\perp^r$ distance from centre to the line is equal to the radius the line touches the circle for all real values of m.

Question on Angle of intersection

**The angle of intersection of the curves $y = x^2$ and $6y = 7 - x^3$ at (1, 1) is**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) None of these.

Solution

(c) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow m_1 = 2$

$6y = 7 - x^3 \Rightarrow \frac{dy}{dx} = -\frac{1}{2} \Rightarrow m_2 = -\frac{1}{2}$

$\therefore \quad m_1 m_2 = -1$ at (1, 1)

$\Rightarrow \quad \theta = \frac{\pi}{2}$.

Question

If $a, x_1, x_2$ are in G.P. with common ratio $r$, and $b, y_1, y_2$ are in G.P. with common ratio $s$ where $s - r = 2$, then the area of the triangle with vertices $(a, b)$, $(x_1, y_1)$ and $(x_2, y_2)$ is

(a) $|ab(r^2 - 1)|$ (b) $ab(r^2 - s^2)$

(c) $ab(s^2 - 1)$ (d) $abrs$

*Ans.* (a)

**Solution** Area of the triangle

$$= \frac{1}{2}\left|\begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix}\right| = \frac{1}{2}|ab(r-1)(s-1)(s-r)|$$

$$= |ab(r-1)(r+1)| = |ab(r^2-1)|$$

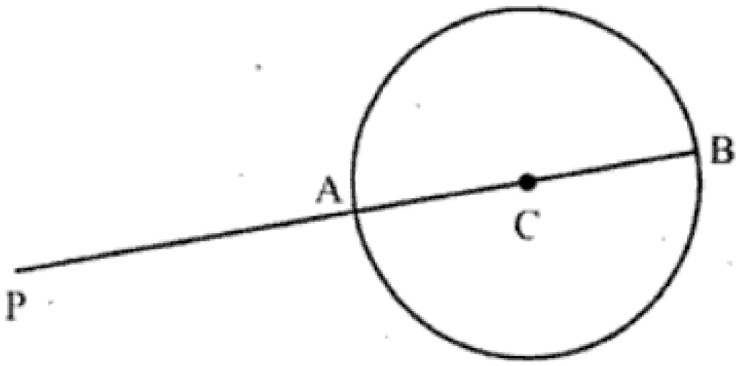
Question

**Let $S \equiv x^2 + y^2 - 4x + 6y - 12 = 0$ and $P = (-13, 17)$ and consider the statements**

A: The nearest point on S from P is (-1,1)
B: The farthest point on S from P is (5,-7), then

**a)** only statement A is true
**b)** only statement B is true
**c)** both the statements A and B are true
**d)** neither statement A nor statement B is true

**Ans (c)**




Here centre, C= (2, -3)

radius

$$= \sqrt{4+9+12} = 5$$

$$CP = \sqrt{(2+13)^2 + (-3-17)^2} = \sqrt{625} = 25 > r$$

$\Rightarrow$ P lies outside the circle.

let A, B be the nearest and farthest points on the circle from P

$\therefore$ PA+AC = CP $\Rightarrow$ PA+5 = 25 $\Rightarrow$ PA = 20

Also

PB=PC+CB $\Rightarrow$ PB = 25+5 25 $\Rightarrow$ PB = 30

Now A divides PC in the ratio

PA:AC=20:5=4:1

$$\Rightarrow A = \left(\frac{4(2)+1(-13)}{4+1}, \frac{4(-13)+1(17)}{4+1}\right)$$

$$= (-1,1)$$

Now B divides PC in the ratio PB : BC =

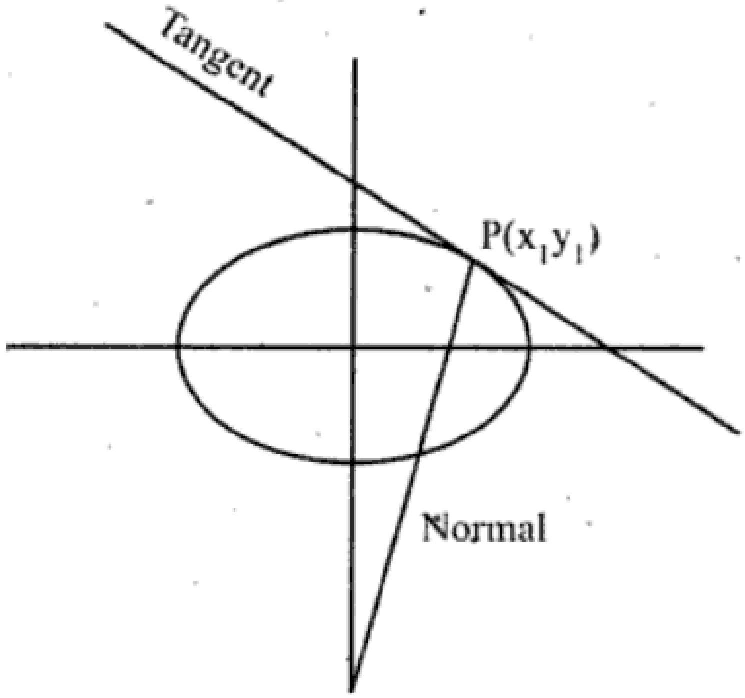
30:5=6:1 externally

$$\therefore \quad B = \left(\frac{6(2)-1(-13)}{6-1}, \frac{6(-3)-1(17)}{6-1}\right)$$

$$= (5,-7)$$

Formulae related to ellipse

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1 y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$




The equation of normal to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1 y_1)$ is $\frac{a^2 x}{x_1}-\frac{b^2 y}{y_1}=a^2-b^2$

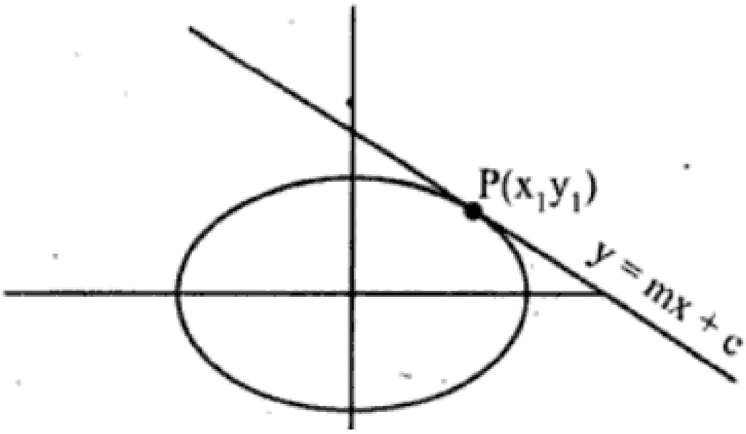
**Note:**

Four normals can be drawn from any point to the ellipse.

## Condition for y = mx + c to be a tangent to the ellipse and points of tangency

The equation of tangent to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1 y_1)$ is



$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ...(1)

Given - mx + y = c ..(2)

(1) and (2) represent the same line

thus $\frac{\frac{x_1}{a^2}}{-m} = \frac{\frac{y_1}{b^2}}{1} = \frac{1}{c}$

$\Rightarrow x_1 = \frac{-a^2 m}{c},\ y_1 = \frac{b^2}{c}$

Since $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

we get, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

$\Rightarrow \frac{a^4 m^2}{c^2 a^2} + \frac{b^4}{c^2 b^2} = 1$

Formulae related to Hyperbola

## Parametric equations of the hyperbola

A point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be represented as $x = a \sec\theta$, $y = b \tan\theta$ in a single parameter $\theta$. These equations are called parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The point $(a \sec\theta, b \tan\theta)$ is simply denoted by $\theta$.

Some important results

i) The equation of the chord joining the points $(a \sec \alpha, b \tan \alpha)$ and $(a \sec \beta, b \tan \beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}.$$

ii) The equation of the tangent at $P(\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1 .$$

iii) The equation of the normal at $P(\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

**iv)** The condition that the line $lx + my + n = 0$ may be a normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

**v)** If P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci S and $S^1$, then $S^1P - SP = 2a$.

**vi)** The locus of point of intersection of perpendicular tangents to an hyperbola

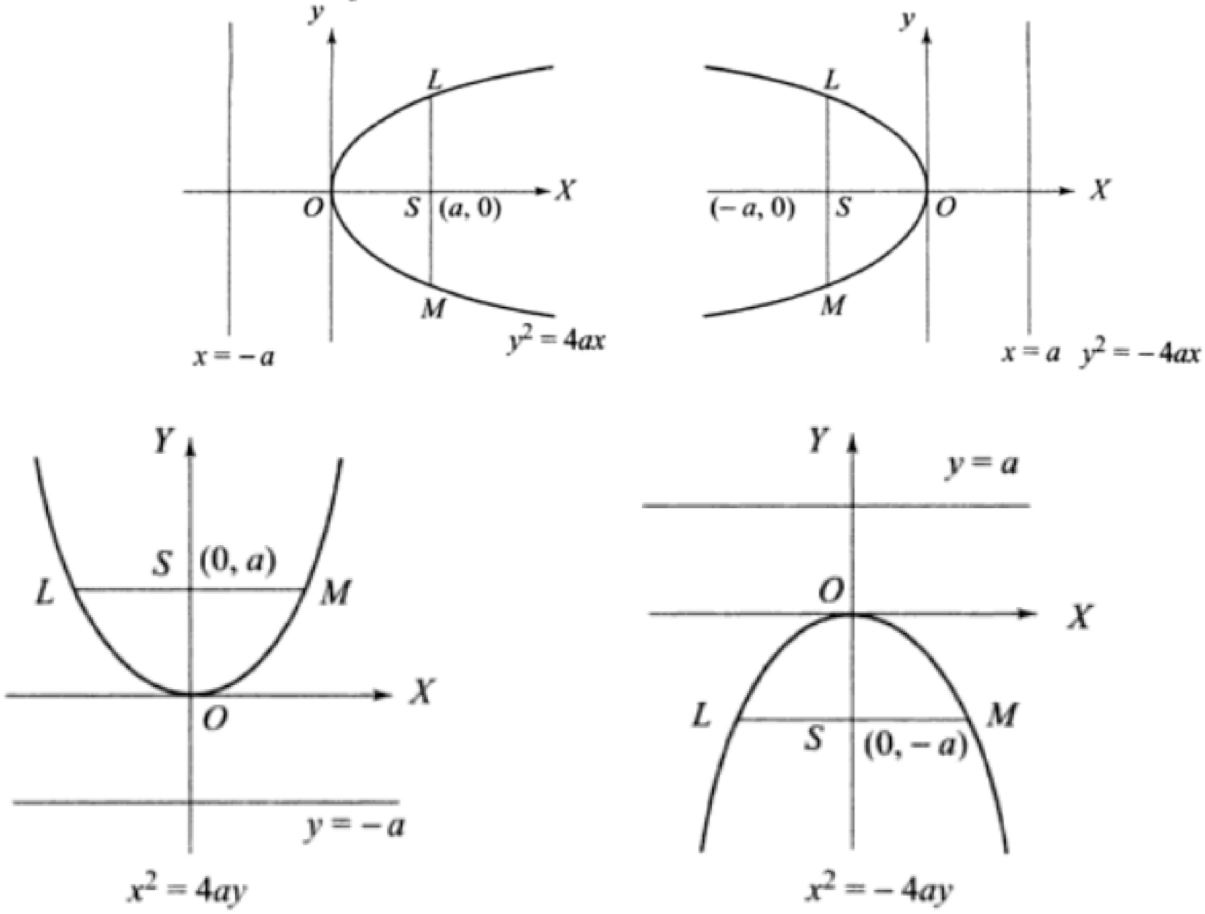
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a circle $x^2 + y^2 = a^2 - b^2$ called director circle of the hyperbola.

**vii)** The locus of the feet of perpendiculars drawn the foci to any tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a circle $x^2 + y^2 = a^2$, called auxiliary circle of the hyperbola.

Parabola

$y^2 = 4ax$ is a *standard form* of the equation of a parabola.
Four standard forms of a parabola are



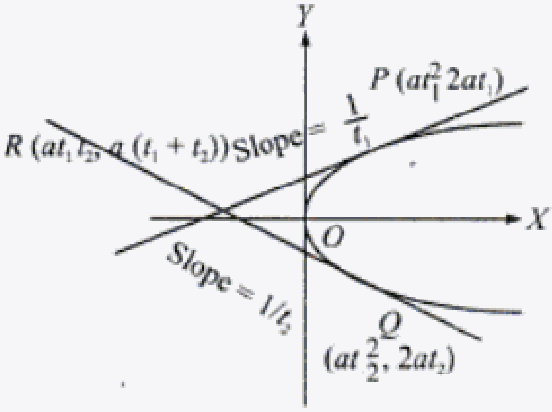

The following terms are used in context of the parabola $y^2 = 4ax$.

1. The point $O(0, 0)$ is the *vertex* of the parabola, and the tangent to the parabola at the vertex is $x = 0$.
2. The line joining the vertex $O$ and the focus $S(a, 0)$ is the *axis* of *the parabola* and its equation is therefore $y = 0$.
3. Any chord of the parabola perpendicular to its axis is called a *double ordinate*.
4. Any chord of the parabola passing through its focus is called a *focal chord.*
5. The focal chord of the parabola perpendicular to its axis is called *its latus rectum*; the length of this latus rectum is therefore $4a$.
6. The points on a parabola, the normals at which are concurrent, are called *co-normal points* of the parabola. If $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ are conormal points of the parabola $y^2 = 4ax$, then $y_1 + y_2 + y_3 = 0$.
7. A line which bisects a system of parallel chords of a parabola is called a *diameter* of the parabola.

The following are some *standard results for the parabola* $y^2 = 4ax$:

1. The *parametric equations* of the parabola or the coordinates of any point on it are $x = at^2$, $y = 2at$.
2. The *tangent* to the parabola at $(x', y')$ is $yy' = 2a(x + x')$ and that at $(at^2, 2at)$ is $ty = x + at^2$.
3. The condition that the line $y = mx + c$ is a tangent to the parabola is $c = a/m$ and the equation of any tangent to it (not parallel to the $y$-axis) is therefore $y = mx + (a/m)$.
4. The *chord of contact* (defined as in circles) of $(x', y')$ w.r.t. the parabola is $yy' = 2a(x + x')$.
5. The *polar* (defined as in circle) of $(x', y')$ w.r.t. the parabola is $yy' = 2a(x + x')$.
6. The *chord with mid-Point* $(x', y')$ of the parabola is $T = S'$, where $T = yy' - 2a(x + x')$ and $S' = y'^2 - 4ax'$.
7. The equation of the *pair of tangents* from $(x', y')$ to the parabola is $T^2 = SS'$. Where $S = y^2 - 4ax$.
8. The *normal* at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^3$. If $m$ is the slope of this normal, then its equation is $y = mx - 2am - am^3$, which is the normal to the parabola at $(am^2, -2am)$.
9. A *diameter* of the parabola is the locus of the middle points of a system of parallel chords of the parabola and the equation of a diameter is $y = 2a/m$ where $m$ is the slope of the parallel chords which are bisected by it.
10. The equation of a chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.
11. If the chord joining the points having parameters $t_1$ and $t_2$ passes through the focus, then $t_1 t_2 = -1$.
12. If the coordinates of one end of a focal chord are $(at^2, 2at)$, then the coordinates of the other end are $(a/t^2, -2a/t)$.
13. For the end of the latus rectum, the values of the parameters $t$ are $\pm 1$.
14. The tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2; 2at_2)$ intersect at $(at_1 t_2, a(t_1 + t_2))$.
15. The tangents at the extremities of any focal chord intersect at right angles on the directrix.
16. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.
17. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.
18. The circle described on any focal chord of a parabola as diameter touches the directrix.

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is




$R \equiv (at_1t_2, a(t_1 + t_2))$.

$\left( \text{i.e.} \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2) \right)$ is the y-coordinate of the point of intersection of tangents at $P$ and $Q$ on the parabola.

The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle $\alpha$ is
$(x + \alpha)^2 \tan^2 \alpha = y^2 - 4ax$

The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at $R$. Then the area of triangle $PQR$ is $\frac{1}{2}a^2(t_1 - t_2)^3$.

If the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, then $ln = am^2$.

If the line $\frac{x}{l} + \frac{y}{m} = 1$ touches the parabola $y^2 = 4a(x + b)$ then $m^2(l + b) + al^2 = 0$.

If the two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at point $P$, whose abscissa is not zero, then the tangent to each curve at $P$, make complementary angle with the $x$-axis.

If the line $x \cos \alpha + y \sin a = p$ touches the parabola $y^2 = 4ax$, then $p \cos \alpha + a \sin^2 \alpha = 0$ and the point of contact is $(a \tan^2\alpha, -2a \tan \alpha)$

Tangents at the extremities of any focul chord of a parabola meet at right angle on the directrix.

Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

If the tangents at the points $P$ and $Q$ on a parabola meet in $T$, then $ST$ is the geometric mean between $SP$ and $SQ$, i.e., $ST^2 = SP \cdot SQ$.

## POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point $(x_1, y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =$ or $< 0$, respectively.

## NUMBER OF TANGENTS DRAWN FROM A POINT TO A PARABOLA

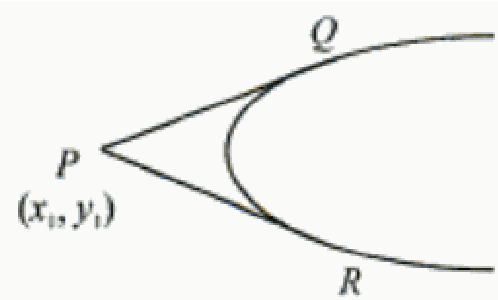
Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

## EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$,

where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1^2 - 4ax_1$

and $T \equiv yy_1 - 2a(x + x_1)$




## EQUATIONS OF NORMAL IN DIFFERENT FORMS

**1. Point Form** The equation of the normal to the parabola $y^2 = 4ax$ at a point $(x_1, y_1)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

**2. Parametric Form** The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3.$$

**3. Slope Form** The equation of normal to the parabola $y^2 = 4ax$ in terms of slope '$m$' is

$$y = mx - 2am - am^3.$$

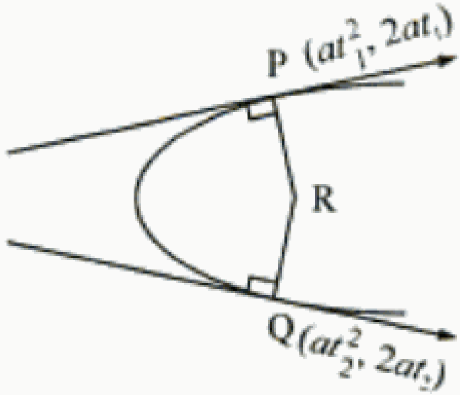
***Note:*** The coordinates of the point of contact are $(am^2, -2am)$.

**Condition for Normality** The line $y = mx + c$ is a normal to the parabola

$$y^2 = 4ax \text{ if } c = -2am - am^3.$$

## POINT OF INTERSECTION OF NORMALS

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

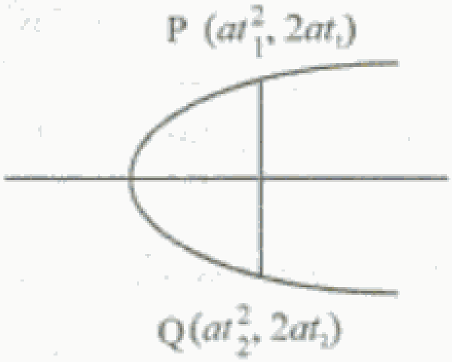



$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)].$$

If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}.$$

Note that $PQ$ is normal to the parabola at $P$ and not at $Q$.



If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$.

## CO-NORMAL POINTS

Any three points on a parabola normals at which pass through a common point are called co-normal points

If three normals are drawn through a point $(h, k)$, then their slopes are the roots of the cubic :

$$k = mh - 2am - am^3$$

(i) The sum of the slopes of the normals at co-normal points is zero, i.e. $m_1 + m_2 + m_3 = 0$.

(ii) The sum of the ordinates of the co-normal points is zero (i.e. $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$).

(iii) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola [the vertices of the triangle formed by the co-normal points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. Thus, $y$-coordinate of the centroid becomes

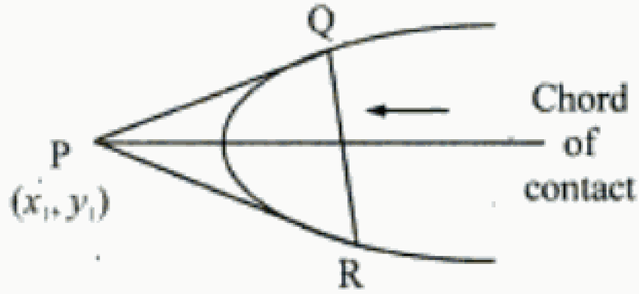
$$\frac{-2a(m_1 + m_2 + m_3)}{3} = \frac{-2a}{3} \times 0 = 0.$$

Hence, the centroid lies on the $x$-axis, i.e. axis of the parabola.]

(iv) If three normals drawn to any parabola $y^2 = 4ax$ from a given point $(h, k)$ be real, then $h > 2a$.

## CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.

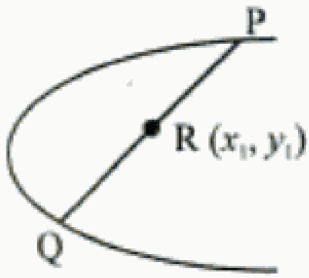



## CHORD WITH A GIVEN MID POINT

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1$$

where $T \equiv yy_1 - 2a(x + x_1)$ and $S_1 \equiv y_1^2 - 4ax$.



Question

If the tangent to the parabola $y^2 = 4ax$ meets the axis in $T$ and tangent at the vertex $A$ in $Y$ and the rectangle TAYG is completed, then the locus of $G$ is

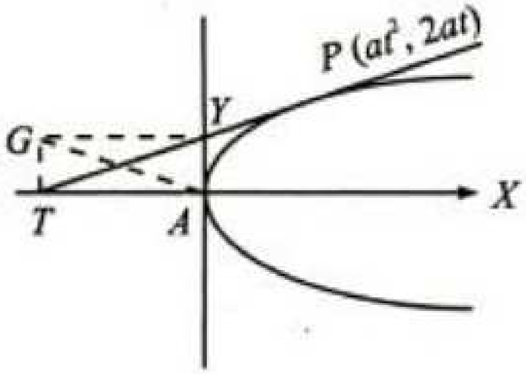
(a) $y^2 + 2ax = 0$ (b) $y^2 + ax = 0$
(c) $x^2 + ay = 0$ (d) none of these

Solution

**(b).** Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$. The equation of tangent at $P$ is $ty = x + at^2$.

Since the tangent meets the axis of parabola in $T$ and tangent at the vertex $A$ in $Y$,

$\therefore$ coordinates of $T$ and $Y$ are $(-at^2, 0)$ and $(0, at)$ respectively.




Let the coordinates of $G$ be $(x_1, y_1)$.
Since TAYG is a rectangle,
$\therefore$ midpoint of diagonals $TY$ and $GA$ is same

$$\Rightarrow \quad \frac{x_1 + 0}{2} = \frac{-at^2 + 0}{2} \text{ and } \frac{y_1 + 0}{2} = \frac{0 + at}{2}$$

$$\Rightarrow \quad x_1 = -at^2 \qquad ...(1)$$

and $$y_1 = at \qquad ...(2)$$

Eliminating $t$ from (1) and (2), we get

$$x_1 = -a\left(\frac{y_1}{a}\right)^2 \Rightarrow y_1^2 + ax_1 = 0.$$

$\therefore$ The locus of $G\ (x_1, y_1)$ is $\boldsymbol{y^2 + ax = 0}$.

Question

Equation of a common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$
(b) $y = 2x + 1$
(c) $2y = x + 8$
(d) $y = x + 2$

*Ans.* (d)

Solution Equation of a tangent at $(at^2, 2at)$ to $y^2 = 8x$ is

$ty = x + at^2$ where $4a = 8$ *i.e.* $a = 2$

$\Rightarrow$ $ty = x + 2t^2$ which intersects the curve $xy = -1$ at the points given by $\dfrac{x(x+2t^2)}{t} = -1$

clearly $t \neq 0$

or $x^2 + 2t^2x + t = 0$ and will be a tangent to the curve if the roots of this quadratic equation are equal, for which $4t^4 - 4t = 0 \Rightarrow t = 0$ or $t = 1$ and an equation of a common tangent is $y = x + 2$.

Question

Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = 12x$

Answer

The given equation is $y^2 = 12x$.

Here, the coefficient of $x$ is positive. Hence, the parabola opens towards the right.

On comparing this equation with $y^2 = 4ax$, we obtain

$4a = 12 \Rightarrow a = 3$

$\therefore$ Coordinates of the focus $= (a, 0) = (3, 0)$

Since the given equation involves $y^2$, the axis of the parabola is the $x$-axis.

Equation of direcctrix, $x = -a$ i.e., $x = -3$ i.e., $x + 3 = 0$

Length of latus rectum $= 4a = 4 \times 3 = 12$

Question

The conic represented by the equation $\sqrt{ax} + \sqrt{by} = 1$ is

(a) ellipse (b) Hyperbola

(c) parabola (d) none of these

Solution

**(c).** The given conic is $\sqrt{ax} + \sqrt{by} = 1$

Squaring both sides,

$$ax + by + 2\sqrt{abxy} = 1$$

or $$ax + by - 1 = -2\sqrt{abxy}.$$

Squaring again, $(ax + by - 1)^2 = 4abxy$

or $$a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0 \quad ...(1)$$

Comparing the equation (1) with the equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

$\therefore \quad A = a^2, H = -ab, B = b^2, G = -a, F = -b, C = 1.$

Then, $\Delta = ABC + 2FGH - AF^2 - BG^2 - CH^2$

$= a^2b^2 - 2a^2b^2 - a^2b^2 - a^2b^2 - a^2b^2$

$= -4a^2b^2 \neq 0$

and $H^2 = a^2b^2 = AB.$

So we have $\Delta \neq 0$ and $H^2 - AB = 0$. Hence the given equation represents a **parabola.**

Question

Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = 6y$

Answer

The given equation is $x^2 = 6y$.

Here, the coefficient of $y$ is positive. Hence, the parabola opens upwards.

On comparing this equation with $x^2 = 4ay$, we obtain

$4a = 6 \Rightarrow a = \frac{3}{2}$

∴Coordinates of the focus = $(0, a)$ = $\left(0, \frac{3}{2}\right)$

Since the given equation involves $x^2$, the axis of the parabola is the $y$-axis.

Equation of directrix, $y = -a$ i.e., $y = -\frac{3}{2}$

Length of latus rectum = $4a$ = 6

Question

Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = -8x$

Answer

The given equation is $y^2 = -8x$.

Here, the coefficient of $x$ is negative. Hence, the parabola opens towards the left.

On comparing this equation with $y^2 = -4ax$, we obtain

$-4a = -8 \Rightarrow a = 2$

∴Coordinates of the focus = $(-a, 0) = (-2, 0)$

Since the given equation involves $y^2$, the axis of the parabola is the $x$-axis.

Equation of directrix, $x = a$ i.e., $x = 2$

Length of latus rectum = $4a$ = 8

Question

Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = -16y$

Answer

The given equation is $x^2 = -16y$.

Here, the coefficient of $y$ is negative. Hence, the parabola opens downwards.

On comparing this equation with $x^2 = -4ay$, we obtain

$-4a = -16 \Rightarrow a = 4$

$\therefore$ Coordinates of the focus = $(0, -a) = (0, -4)$

Since the given equation involves $x^2$, the axis of the parabola is the $y$-axis.

Equation of directrix, $y = a$ i.e., $y = 4$

Length of latus rectum = $4a = 16$

Question

Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = 10x$

Answer

The given equation is $y^2 = 10x$.

Here, the coefficient of $x$ is positive. Hence, the parabola opens towards the right.

On comparing this equation with $y^2 = 4ax$, we obtain

$4a = 10 \Rightarrow a = \frac{5}{2}$

$\therefore$ Coordinates of the focus = $(a, 0) = \left(\frac{5}{2}, 0\right)$

Since the given equation involves $y^2$, the axis of the parabola is the $x$-axis.

Equation of directrix, $x = -a$, i.e., $x = -\frac{5}{2}$

Length of latus rectum = $4a = 10$

Question

Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = -9y$

Answer

The given equation is $x^2 = -9y$.

Here, the coefficient of $y$ is negative. Hence, the parabola opens downwards.

On comparing this equation with $x^2 = -4ay$, we obtain

$$-4a = -9 \Rightarrow b = \frac{9}{4}$$

∴Coordinates of the focus = $(0, -a) = \left(0, -\frac{9}{4}\right)$

Since the given equation involves $x^2$, the axis of the parabola is the $y$-axis.

Equation of directrix, $y = a$, i.e., $y = \frac{9}{4}$

Length of latus rectum = $4a$ = 9

Question

Find the equation of the parabola that satisfies the following conditions: Focus (6, 0); directrix $x = -6$

Answer

Focus (6, 0); directrix, $x = -6$

Since the focus lies on the $x$-axis, the $x$-axis is the axis of the parabola.

Therefore, the equation of the parabola is either of the form $y^2 = 4ax$ or $y^2 = - 4ax$.

It is also seen that the directrix, $x = -6$ is to the left of the $y$-axis, while the focus (6, 0) is to the right of the $y$-axis. Hence, the parabola is of the form $y^2 = 4ax$.

Here, $a = 6$

Thus, the equation of the parabola is $y^2 = 24x$.

Question

Find the equation of the parabola that satisfies the following conditions: Focus (0, −3); directrix $y = 3$

Answer

Focus = (0, −3); directrix $y = 3$

Since the focus lies on the $y$-axis, the $y$-axis is the axis of the parabola.

Therefore, the equation of the parabola is either of the form $x^2 = 4ay$ or $x^2 = -4ay$.

It is also seen that the directrix, $y = 3$ is above the $x$-axis, while the focus (0, −3) is below the $x$-axis. Hence, the parabola is of the form $x^2 = -4ay$.

Here, $a = 3$

Thus, the equation of the parabola is $x^2 = -12y$.

Question

Find the equation of the parabola that satisfies the following conditions: Vertex (0, 0); focus (3, 0)

Answer

Vertex (0, 0); focus (3, 0)

Since the vertex of the parabola is (0, 0) and the focus lies on the positive $x$-axis, $x$-axis is the axis of the parabola, while the equation of the parabola is of the form $y^2 = 4ax$.

Since the focus is (3, 0), $a = 3$.

Thus, the equation of the parabola is $y^2 = 4 \times 3 \times x$, i.e., $y^2 = 12x$

Question

Find the equation of the parabola that satisfies the following conditions: Vertex (0, 0) focus (−2, 0)

Answer

Vertex (0, 0) focus (−2, 0)

Since the vertex of the parabola is (0, 0) and the focus lies on the negative $x$-axis, $x$-axis is the axis of the parabola, while the equation of the parabola is of the form $y^2 = -4ax$.

Since the focus is (−2, 0), $a = 2$.

Thus, the equation of the parabola is $y^2 = -4(2)x$, i.e., $y^2 = -8x$

Question

Find the equation of the parabola that satisfies the following conditions: Vertex (0, 0) passing through (2, 3) and axis is along $x$-axis

Answer

Since the vertex is (0, 0) and the axis of the parabola is the $x$-axis, the equation of the parabola is either of the form $y^2 = 4ax$ or $y^2 = -4ax$.

The parabola passes through point (2, 3), which lies in the first quadrant.

Therefore, the equation of the parabola is of the form $y^2 = 4ax$, while point (2, 3) must satisfy the equation $y^2 = 4ax$.

$$\therefore 3^2 = 4a(2) \Rightarrow a = \frac{9}{8}$$

Thus, the equation of the parabola is

$$y^2 = 4\left(\frac{9}{8}\right)x$$

$$y^2 = \frac{9}{2}x$$

$$2y^2 = 9x$$

Question

Find the equation of the parabola that satisfies the following conditions: Vertex (0, 0), passing through (5, 2) and symmetric with respect to $y$-axis

Answer

Since the vertex is (0, 0) and the parabola is symmetric about the $y$-axis, the equation of the parabola is either of the form $x^2 = 4ay$ or $x^2 = -4ay$.

The parabola passes through point (5, 2), which lies in the first quadrant.

Therefore, the equation of the parabola is of the form $x^2 = 4ay$, while point (5, 2) must satisfy the equation $x^2 = 4ay$.

$$\therefore (5)^2 = 4 \times a \times 2 \Rightarrow 25 = 8a \Rightarrow a = \frac{25}{8}$$

Thus, the equation of the parabola is

$$x^2 = 4\left(\frac{25}{8}\right)y$$

$$2x^2 = 25y$$

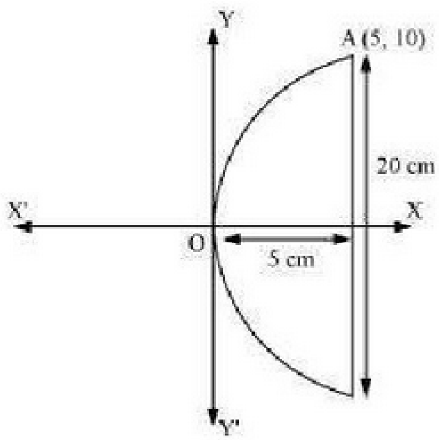
Question

If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

Answer

The origin of the coordinate plane is taken at the vertex of the parabolic reflector in such a way that the axis of the reflector is along the positive $x$-axis.

This can be diagrammatically represented as




The equation of the parabola is of the form $y^2 = 4ax$ (as it is opening to the right).

Since the parabola passes through point A (10, 5), $10^2 = 4a(5)$

$\Rightarrow 100 = 20a$

$$\Rightarrow a = \frac{100}{20} = 5$$

Therefore, the focus of the parabola is $(a, 0) = (5, 0)$, which is the mid-point of the diameter.

Hence, the focus of the reflector is at the mid-point of the diameter.

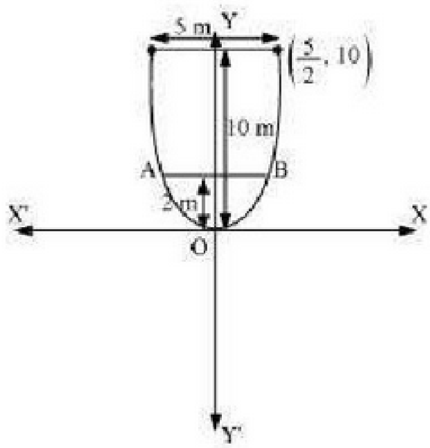
Question

An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

Answer

The origin of the coordinate plane is taken at the vertex of the arch in such a way that its vertical axis is along the positive $y$-axis.

This can be diagrammatically represented as




The equation of the parabola is of the form $x^2 = 4ay$ (as it is opening upwards).

It can be clearly seen that the parabola passes through point $\left(\frac{5}{2}, 10\right)$.

$$\left(\frac{5}{2}\right)^2 = 4a(10)$$

$$\Rightarrow a = \frac{25}{4\times4\times10} = \frac{5}{32}$$

Therefore, the arch is in the form of a parabola whose equation is $x^2 = \frac{5}{8}y$.

When y = 2 m, $x^2 = \frac{5}{8}\times 2$

$$\Rightarrow x^2 = \frac{5}{4}$$

$$\Rightarrow x = \sqrt{\frac{5}{4}} \text{ m}$$

$$\therefore \text{AB} = 2\times\sqrt{\frac{5}{4}} \text{ m} = 2\times1.118 \text{ m (approx.)} = 2.23 \text{ m (approx.)}$$

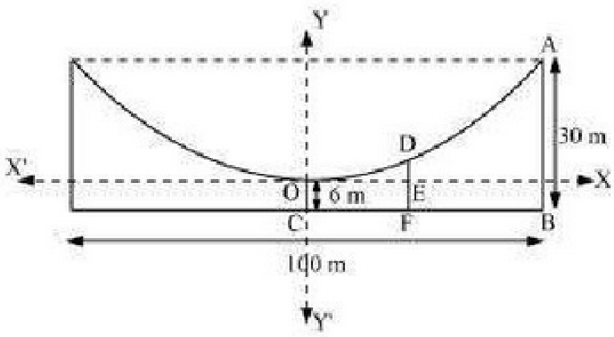
Hence, when the arch is 2 m from the vertex of the parabola, its width is approximately 2.23 m.

Question

The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Answer

The vertex is at the lowest point of the cable. The origin of the coordinate plane is taken as the vertex of the parabola, while its vertical axis is taken along the positive $y$-axis. This can be diagrammatically represented as




Here, AB and OC are the longest and the shortest wires, respectively, attached to the cable.

DF is the supporting wire attached to the roadway, 18 m from the middle.

Here, AB = 30 m, OC = 6 m, and $BC = \frac{100}{2} = 50 \text{ m}$.

The equation of the parabola is of the form $x^2 = 4ay$ (as it is opening upwards).

The coordinates of point A are (50, 30 − 6) = (50, 24).

Since A (50, 24) is a point on the parabola,

$$(50)^2 = 4a(24)$$

$$\Rightarrow a = \frac{50 \times 50}{4 \times 24} = \frac{625}{24}$$

∴Equation of the parabola, $x^2 = 4 \times \frac{625}{24} \times y$ or $6x^2 = 625y$

The $x$-coordinate of point D is 18.

Hence, at $x = 18$,

$$6(18)^2 = 625y$$

$$\Rightarrow y = \frac{6 \times 18 \times 18}{625}$$

$$\Rightarrow y = 3.11 \text{ (approx)}$$

∴DE = 3.11 m

DF = DE + EF = 3.11 m + 6 m = 9.11 m

Thus, the length of the supporting wire attached to the roadway 18 m from the middle is approximately 9.11 m.

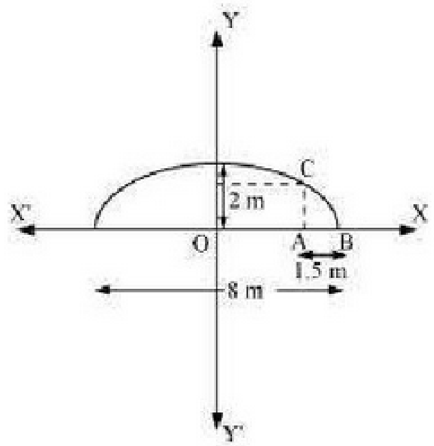
Question

An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

Answer

Since the height and width of the arc from the centre is 2 m and 8 m respectively, it is clear that the length of the major axis is 8 m, while the length of the semi-minor axis is 2 m.

The origin of the coordinate plane is taken as the centre of the ellipse, while the major axis is taken along the $x$-axis. Hence, the semi-ellipse can be diagrammatically represented as



The equation of the semi-ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ y\geq 0$, where $a$ is the semi-major axis

Accordingly, $2a = 8 \Rightarrow a = 4$

$b = 2$

Therefore, the equation of the semi-ellipse is $\frac{x^2}{16}+\frac{y^2}{4}=1,\ y\geq 0$ ...(1)

Let A be a point on the major axis such that AB = 1.5 m.

Draw AC⊥ OB.

OA = (4 – 1.5) m = 2.5 m

The $x$-coordinate of point C is 2.5.

On substituting the value of $x$ with 2.5 in equation (1), we obtain

$$\frac{(2.5)^2}{16}+\frac{y^2}{4}=1$$

$$\Rightarrow \frac{6.25}{16}+\frac{y^2}{4}=1$$

$$\Rightarrow y^2=4\left(1-\frac{6.25}{16}\right)$$

$$\Rightarrow y^2=4\left(\frac{9.75}{16}\right)$$

$$\Rightarrow y^2=2.4375$$

$$\Rightarrow y=1.56 \quad \text{(approx.)}$$

∴AC = 1.56 m

Thus, the height of the arch at a point 1.5 m from one end is approximately 1.56 m.

Question

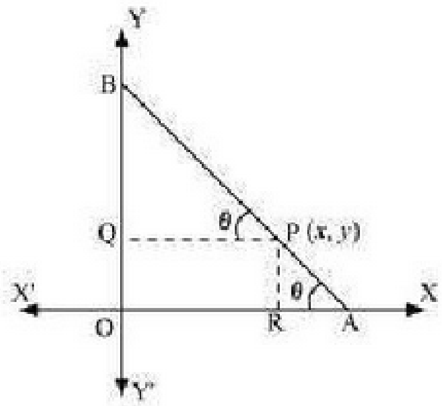
A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the $x$-axis.

Answer

Let AB be the rod making an angle $\theta$ with OX and P ($x$, $y$) be the point on it such that AP = 3 cm.

Then, PB = AB − AP = (12 − 3) cm = 9 cm [AB = 12 cm]

From P, draw PQ⊥OY and PR⊥OX.




In ΔPBQ, $\cos\theta = \frac{PQ}{PB} = \frac{x}{9}$

In ΔPRA, $\sin\theta = \frac{PR}{PA} = \frac{y}{3}$

Since, $\sin^2\theta + \cos^2\theta = 1$,

$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{9}\right)^2 = 1$$

Or, $\frac{x^2}{81} + \frac{y^2}{9} = 1$

Thus, the equation of the locus of point P on the rod is $\frac{x^2}{81} + \frac{y^2}{9} = 1$

Question

Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

Answer

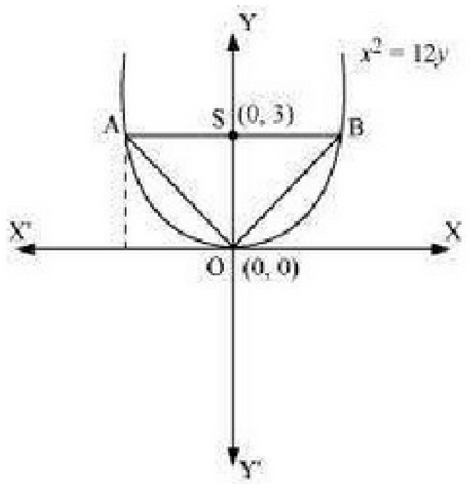
The given parabola is $x^2 = 12y$.

On comparing this equation with $x^2 = 4ay$, we obtain $4a = 12 \Rightarrow a = 3$

∴The coordinates of foci are S $(0, a)$ = S $(0, 3)$

Let AB be the latus rectum of the given parabola.

The given parabola can be roughly drawn as




At $y = 3$, $x^2 = 12(3) \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

∴The coordinates of A are $(-6, 3)$, while the coordinates of B are $(6, 3)$.

Therefore, the vertices of ΔOAB are O $(0, 0)$, A $(-6, 3)$, and B $(6, 3)$.

$$\text{Area of } \Delta OAB = \frac{1}{2}\left|0(3-3)+(-6)(3-0)+6(0-3)\right| \text{ unit}^2$$

$$= \frac{1}{2}\left|(-6)(3)+6(-3)\right| \text{ unit}^2$$

$$= \frac{1}{2}\left|-18-18\right| \text{ unit}^2$$

$$= \frac{1}{2}\left|-36\right| \text{ unit}^2$$

$$= \frac{1}{2} \times 36 \text{ unit}^2$$
$$= 18 \text{ unit}^2$$

Thus, the required area of the triangle is 18 unit$^2$.

Question

The tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at $Q$ and $R$, the coordinates of the mid-point of $QR$ are

(a) $(x_1 - a, y_1 + b)$ (b) $(x_1, y_1)$
(c) $(x_1 + b, y_1 + a)$ (d) $(x_1 - b, y_1 - b)$

*Ans.* (b)

**Solution** Equation of the tangent at $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

or $2ax - y_1y + 2ax_1 = 0$ (i)

If $M(h, k)$ is the mid-point of $QR$, then equation of $QR$ a chord of the parabola $y^2 = 4a(x + b)$ in terms of its mid-point is $ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$

(using $T = S'$) or $2ax - ky + k^2 - 2ah = 0$ (ii)

Since (i) and (ii) represent the same line, we have

$$\frac{2a}{2a} = \frac{y_1}{k} = \frac{2ax_1}{k^2 - 2ah} \Rightarrow k = y_1 \text{ and } k^2 - 2ah = 2ax_1$$

$\Rightarrow$ $y_1^2 - 2ah = 2ax_1 \Rightarrow 4ax_1 - 2ax_1 = 2ah$

(as $P(x_1, y_1)$ lies on the parabola $y^2 = 4ax$)

$\Rightarrow$ $h = x_1$ so that $h = x_1$, $k = y_1$ and the mid point of $QR$ is $(x_1, y_1)$

Question

The focus of the parabola $4y^2 + 12x - 20y + 67 = 0$ is

(a) $(-7/2, 5/2)$ (b) $(-3/4, 5/2)$
(c) $(-17/4, 5/2)$ (d) $(5/2, -3/4)$

*Ans.* (c)

**Solution** The given equation of the parabola can be written as

$$y^2 - 5y = -3x - 67/4 \Rightarrow (y - 5/2)^2 = -3(x + 7/2)$$

$\Rightarrow$ $Y^2 = 4aX$ where $Y = y - 5/2$, $X = x + 7/2$ and $a = -3/4$

The focus of $Y^2 = 4aX$ is $(X, Y) = (a, 0) = (-3/4, 0)$

$\Rightarrow$ $x + 7/2 = -3/4,\ y - 5/2 = 0 \Rightarrow x = -17/4,\ y = 5/2$

Therefore, required focus is $(-17/4, 5/2)$

Question

The point of intersection of the tangents to the parabola $y^2 = 4x$ at the points where the circle $(x-3)^2 + y^2 = 9$ meets the parabola, other than the origin, is

(a) $(-2, 0)$ (b) $(1, 0)$ (c) $(0, 0)$ (d) $(-1, -1)$

*Ans.* (a)

**Solution** The circle meets the parabola at points given by $(x-3)^2 + 4x = 9$

$\Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, x = 2$. But $x = 0$ gives the origin so we take $x = 2$ and $y = \pm 2\sqrt{2}$. Equation of the tangents to the parabola at $(2, 2\sqrt{2})$ and $(2, -2\sqrt{2})$ are respectively.

$$y(2\sqrt{2}) = 2(x+2) \text{ and } y(-2\sqrt{2}) = 2(x+2)$$

Solving these we get $y = 0$ and $x = -2$.

Question

If $P, Q, R$ are three points on a parabola $y^2 = 4ax$ whose ordinates are in geometrical progression, then the tangents at $P$ and $R$ meet on

(a) the line through $Q$ parallel to $x$-axis
(b) the line through $Q$ parallel to $y$-axis
(c) the line joining $Q$ to the vertex
(d) the line joining $Q$ to the focus.

*Ans.* (b)

**Solution** Let the coordinates of $P, Q, R$ be $(at_i^2, 2at_i)$ $i = 1, 2, 3$ having ordinates in G.P. So that $t_1, t_2, t_3$ are also in G.P. i.e. $t_1t_3 = t_2^2$. Equations of the tangents at $P$ and $R$ are

$$t_1 y = x + at_1^2 \text{ and } t_3 y = x + at_3^2, \text{ which intersect}$$

at the point $\dfrac{x + at_1^2}{t_1} = \dfrac{x + at_3^2}{t_3} \quad \Rightarrow \quad x = at_1t_3 = at_2^2$

which is a line through $Q$ parallel to $Y$-axis.

Question

Equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is

(a) $x = -1$ (b) $x = 1$ (c) $x = -3/2$ (d) $x = 3/2$

*Ans.* (d)

**Solution** Given equation can be written as

$$(y + 2)^2 = -4x + 2 = -4(x - 1/2)$$

which is of the form $Y^2 = 4aX$

where $Y = y + 2, X = x - 1/2, a = -1$

The directrix of the parabola $Y^2 = 4aX$ is $X = -a$

$\Rightarrow$ $x - 1/2 = -(-1)$ $\Rightarrow$ $x = 3/2$

is the equation of the directrix of the given parabola

Question

The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

(a) $x = -a$ (b) $x = -a/2$ (c) $x = 0$ (d) $x = a/2$

*Ans.* (c)

**Solution** The focus of the parabola $y^2 = 4ax$ is $S(a, 0)$, let $P(at^2, 2at)$ be any point on the parabola then coordinates of the mid-point of $SP$ are given by

$$x = \frac{a(t^2+1)}{2}, \quad y = \frac{2at+0}{2}$$

Eliminating '$t$' we get the locus of the mid-point

as $y^2 = 2ax - a^2$ or $y^2 = 2a(x - a/2)$ (1)

which is a parabola of the form $Y^2 = 4AX$ (2)

Where $Y = y, X = x - a/2$ and $A = a/2$

Equation of the directrix of (2) is $X = -A$

So equation the directrix of (1) is $x - a/2 = -a/2$

$\Rightarrow$ $x = 0$

Question

If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then it touches the parabola

(a) $y^2 = -36x$ (b) $y^2 = -12x$ (c) $y^2 = -9x$ (d) none of these

*Ans.* (a)

**Solution** Since $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$, taking $a = 3$ and $m = -1$ we get

$y = -x - 2(3)(-1) - 3(-1)^3 \Rightarrow x + y = 9$ is a normal to the parabola $y^2 = 12x$. Suppose it touches the parabola $y^2 = 4ax$.

Equation of a tangent to the parabola $y^2 = 4ax$ is

$$y = mx + a/m$$

If it represents the line $x + y = 9$, then

$$m = -1 \text{ and } a/m = 9 \Rightarrow a = -9$$

So an equation of the required parabola is

$$y^2 = 4(-9)x \text{ or } y^2 = -36x$$

Question

Equation of a common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$

(c) $2y = x + 8$ (d) $y = x + 2$

*Ans.* (d)

**Solution** Equation of a tangent at $(at^2, 2at)$ to $y^2 = 8x$ is

$$ty = x + at^2 \text{ where } 4a = 8 \text{ i.e. } a = 2$$

$\Rightarrow$ $ty = x + 2t^2$ which intersects the curve $xy = -1$ at the points given by $\frac{x(x+2t^2)}{t} = -1$ clearly $t \neq 0$

or $x^2 + 2t^2x + t = 0$ and will be a tangent to the curve if the roots of this quadratic equation are equal, for which $4t^4 - 4t = 0 \Rightarrow t = 0$ or $t = 1$ and an equation of a common tangent is $y = x + 2$.

Question

If the normal chord at a point '$t$' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex, then the value of $t$ is

(a) 4 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) 1

*Ans.* (c)

**Solution** Equation of the normal at '$t$' to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3 \quad \text{(i)}$$

The joint equation of the lines joining the vertex (origin) to the points of intersection of the parabola and the line (i) is

$$y^2 = 4ax\left[\frac{y+tx}{2at+at^3}\right]$$

$\Rightarrow \quad (2t + t^3)y^2 = 4x(y + tx)$

$\Rightarrow 4t\,x^2 - (2t + t^3)\,y^2 + 4xy = 0$

Since these lines are at right angles co-efficient of $x^2$ + coefficient of $y^2 = 0$

$\Rightarrow \quad 4t - 2t - t^3 = 0 \quad \Rightarrow \quad t^2 = 2$

For $t = 0$, the normal line is $y = 0$, i.e. axis of the parabola which passes through the vertex (0, 0).

Question

$AB$ is a chord of the parabola $y^2 = 4ax$ with the end $A$ at the vertex of the given parabola. $BC$ is drawn perpendiculars to $AB$ meeting the axis of the parabola at $C$. The projection of $BC$ on this axis is

(a) $a$ (b) $2a$ (c) $4a$ (d) $8a$

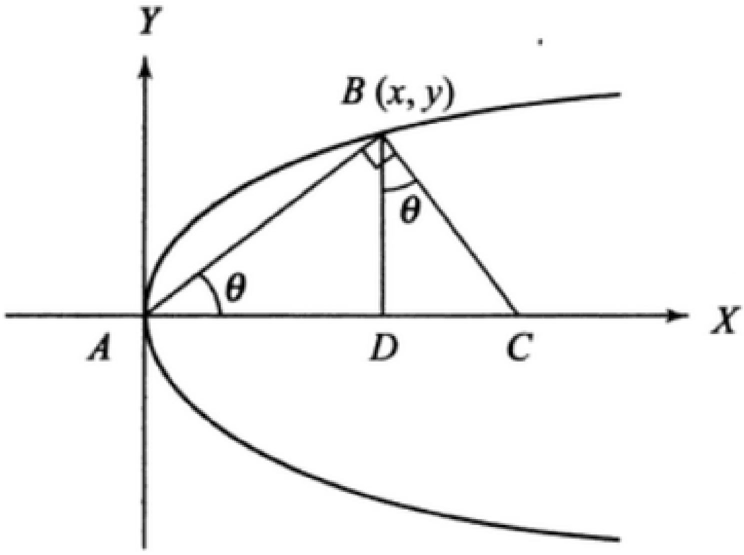
*Ans.* (c)

**Solution** Draw $BD$ perpendicular to the axis of the parabola. Let the coordinates of $B$ be $(x, y)$ then slope of $AB$ is given by

$$\tan\theta = y/x$$

Projection of $BC$ on the axis of the parabola is $DC = BD \tan\theta$

$$= y\,(y/x) = y^2/x = 4ax/x = 4a$$




Question

The slopes of the normals to the parabola $y^2 = 4ax$ intersecting at a point on the axis of the parabola at a distance $4a$ from its vertex are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

*Ans.* (a)

Solution Equation of any normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3.$$

If it passes through $(4a, 0)$, the point on the axis $y = 0$, at a distance $4a$ from the vertex = (0,0) then $m = 0, \pm\sqrt{2}$

Therefore the slopes of the required normals are

$-\sqrt{2}$, 0, $\sqrt{2}$; which are in A.P.

Question

If the focus of a parabola divides a focal chord of the parabola in segments of length 3 and 2, the length of the latus rectum of the parabola is

(a) 3/2 (b) 6/5 (c) 12/5 (d) 24/5

*Ans.* (d)

**Solution** Let $y^2 = 4ax$ be the equation of the parabola, then the focus is $S(a, 0)$. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be vertices of a focal chord of the parabola, then $t_1 t_2 = -1$. Let $SP = 3$, $SQ = 2$

$$SP = \sqrt{a^2(1-t_1^2)+4a^2t_1^2} = a(1 + t_1^2) = 3 \quad \text{(i)}$$

and $$SQ = a\left(1+\frac{1}{t_1^2}\right) = 2 \quad \text{(ii)}$$

From (*i*) and (ii) we get $t_1^2 = 3/2$ and $a = 6/5$

Hence the length of the latus rectum = 24/5.

Question

Equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the $x$-axis is

(a) $\sqrt{3}\ y = 3x + 1$ (b) $\sqrt{3}\ y = -(x + 3)$

(c) $\sqrt{3}\ y = x + 3$ (d) $\sqrt{3}\ y = -(3x + 1)$

*Ans.* (c)

**Solution** Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + 1/m$.
It will touch the circle $(x-3)^2 + y^2 = 9$ whose centre is (3, 0) and radius is 3 if

$$\left|\frac{0 + m(3) + (1/m)}{\sqrt{1+m^2}}\right| = 3$$

or if $$(3m + 1/m)^2 = 9(1 + m^2)$$

or if $$9m^2 + 6 + 1/m^2 = 9 + 9m^2$$

or if $$m^2 = 1/3 \text{ i.e. } m = \pm 1/\sqrt{3}$$

As the tangent is above the $x$-axis, we take $m = 1/\sqrt{3}$ and thus the required equation is $\sqrt{3}\ y = x + 3$.

Question

$P$ is a point on the parabola whose ordinate equals its abscissa. A normal is drawn to the parabola at $P$ to meet it again at $Q$. If $S$ is the focus of the parabola then the product of the slopes of $SP$ and $SQ$ is

(a) $-1$ (b) $1/2$ (c) $1$ (d) $2$

*Ans.* (a)

**Solution** Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$, then $at^2 = 2at \Rightarrow t = 2$ and thus the coordinates of $P$ are $(4a, 4a)$.

Equation of the normal at $P$ is $y = -tx + 2at + at^3$

$$\Rightarrow \quad y = -2x + 4a + 8a \Rightarrow 2x + y = 12a \quad \text{(i)}$$

which meets the parabola $y^2 = 4ax$ at points given by

$$y^2 = 2a(12a - y) \Rightarrow y^2 + 2ay - 24a^2 = 0$$

$$\Rightarrow \quad y = 4a \text{ or } y = -6a$$

$y = 4a$ corresponds to the point $P$ and $y = -6a \Rightarrow x = 9a$ from (i)

So that the coordinates of $Q$ are $(9a, -6a)$. Since the coordinate of the focus $S$ are $(a, 0)$, slope $SP = 4/3$ and slope of $SQ = -6/8$. Product of the slopes $= -1$.

Question

The point of intersection of the tangents to the parabola $y^2 = 4x$ at the points where the circle $(x-3)^2 + y^2 = 9$ meets the parabola, other than the origin, is

(a) $(-2, 0)$ (b) $(1, 0)$ (c) $(0, 0)$ (d) $(-1, -1)$

*Ans.* (a)

**Solution** The circle meets the parabola at points given by $(x-3)^2 + 4x = 9$ $\Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, x = 2$. But $x = 0$ gives the origin so we take $x = 2$ and $y = \pm 2\sqrt{2}$. Equation of the tangents to the parabola at $(2, 2\sqrt{2})$ and $(2, -2\sqrt{2})$ are respectively.

$$y(2\sqrt{2}) = 2(x+2) \text{ and } y(-2\sqrt{2}) = 2(x+2)$$

Solving these we get $y = 0$ and $x = -2$.

Question

Equation of the directrix of the parabola whose focus is (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is

(a) $x - y = 0$ (b) $x - y - 1 = 0$
(c) $x - y + 2 = 0$ (d) $x + y - 1 = 0$

*Ans.* (c)

**Solution** Since the directrix is parallel to the tangent at the vertex, let the equation of the directrix be.

$$x - y + \lambda = 0$$

But the distance between the focus and directrix is twice the distance between the focus and the tangent at the vertex.

Therefore $\frac{0+0+\lambda}{\sqrt{1+1}} = 2 \times \frac{0-0+1}{\sqrt{1+1}}$

$\because$ focus lies on the same side of the directrix and the tangent at the vertex of the parabola.

$\Rightarrow$ $\lambda = 2$, and the required equation is $x - y + 2 = 0$

Question

The common tangents to the circle $x^2 + y^2 = a^2/2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola

(a) $x^2 = 4ay$ (b) $x^2 = -4ay$
(c) $y^2 = -4ax$ (d) $y^2 = 4a(x + a)$

*Ans.* (c)

**Solution** Equation of a tangent to the parabola $y^2 = 4ax$ is $y = mx + a/m$.
If it touches the circle $x^2 + y^2 = a^2/2$

$$\frac{a}{m} = \left(\frac{a}{\sqrt{2}}\right)\sqrt{1+m^2} \quad \Rightarrow \quad 2 = m^2(1+m^2)$$

$\Rightarrow$ $m^4 + m^2 - 2 = 0 \Rightarrow (m^2 - 1)(m^2 + 2) = 0$

$\Rightarrow$ $m^2 = 1 \Rightarrow m = \pm 1$

Hence the common tangents are $y = x + a$ and $y = -x - a$ which intersect at the point $(-a, 0)$ which is the focus of the parabola $y^2 = -4ax$.

Question

Let $P$ the point (1, 0) and $Q$ a point on the locus $y^2 = 8x$. The locus of mid-point of $PQ$ is

(a) $x^2 + 4y + 2 = 0$
(b) $x^2 - 4y + 2 = 0$
(c) $y^2 - 4y + 2 = 0$
(d) $y^2 + 4x + 2 = 0$

*Ans.* (c)

**Solution** Let the coordinates $Q$ be $(2t^2, 4t)$ and of the mid-point of $PQ$ be $(h, k)$ the $h = \frac{2t^2 + 1}{2}$ and $k = \frac{4t - 0}{2}$. Eliminating '$t$' we get

$h = (k/2)^2 + 1/2$ and the locus of $(h, k)$ is $y^2 - 4x + 2 = 0$

Question

If the tangents to the parabola $y^2 = 4ax$ at $(x_1, y_1)$ and $(x_2, y_2)$ intersect at $(x_3, y_3)$, then

(a) $x_1, x_2, x_3$ are in G. P
(b) $x_1, x_2, x_3$ are in A. P
(c) $y_1, y_2, y_3$ are in G.P
(d) $y_1, y_2, y_3$ are in A. P

Solution

**(a, b).** Let $(x_1, y_1) \equiv (at_1^2, 2at_1)$

and $(x_2, y_2) \equiv (at_2^2, 2at_2)$.

Then, $(x_3, y_3) = [at_1t_2, a(t_1 + t_2)]$

$\therefore \quad x_1x_2 = at_1^2 \cdot at_2^2 = (at_1t_2)^2 = x_3^2$

and $y_3 = a(t_1 + t_2) = \frac{1}{2}(y_1 + y_2)$

$\therefore$ **$x_1, x_2, x_3$ are in G. P and $y_1, y_2, y_3$ are in A.P.**

Question

The locus of the vertices of the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a \text{ is}$$

(a) $xy = 64/105$
(b) $xy = 105/64$
(c) $xy = 3/4$
(d) $xy = 35/16$

*Ans.* (b)

**Solution** Equation of the parabola can be written as

$$\frac{y}{a} = \left(\frac{ax}{\sqrt{3}} + \frac{\sqrt{3}}{4}\right)^2 - \frac{3}{16} - 2$$

or $$\left(x + \frac{3}{4a}\right)^2 = \frac{a^2}{3a}\left(y + \frac{35}{16}a\right)$$

vertex is $x = -3/4a$, $y = -35a/16$

Locus of the vertex is $xy = 105/64$.

Question

The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then

(a) $t_2 = -t_1 + 2/t_1$
(b) $t_2 = t_1 - 2/t_1$
(c) $t_2 = t_1 + 2/t_1$
(d) $t_2 = -t_1 - 2/t_1$

*Ans.* (d)

**Solution** Slope of the line joining the given points is $\frac{2}{t_1 + t_2}$ and the slope of the tangent at $(b\,t_1^2, 2b\,t_1)$ is $1/t_1$.

So $$\frac{2}{t_1 + t_2} \times \frac{1}{t_1} = -1 \Rightarrow t_2 = -t_1 - \frac{2}{t_1}.$$

Question

A circle has its centre at the vertex of the parabola $x^2 = 4y$ and the circle cuts the parabola at the ends of its latus rectum. The equation of the circle is

(a) $x^2 + y^2 = 5$ (b) $x^2 + y^2 = 4$
(c) $x^2 + y^2 = 1$ (d) none of these

Solution

**(a).** Coordinates of the vertex of the parabola $x^2 = 4y$ are (0, 0) and the ends of latus rectum are (2, 1) and (– 2, 1).

∴ Centre of the circle is (0, 0) and radius of the circle is

$$= \sqrt{(2)^2 + (1)^2} = \sqrt{5}.$$

∴ Equation of circle is

$$x^2 + y^2 = 5.$$

Question

The locus of the point of intersection of the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ which are at right angles is

(a) a circle (b) a parabola (c) an ellipse (d) a hyperbola

*Ans.* (a)

**Solution** Equation of the tangent to the given ellipse with slope $m$ is

$$y = mx + \sqrt{a^2m^2 + b^2} \qquad \text{(i)}$$

and the equation of tangent perpendicular to (i) is

$$my + x = \sqrt{a^2 + b^2m^2} \qquad \text{(ii)}$$

Squaring and adding (i) and (ii) to eliminate $m$, we get

$$(y - mx)^2 + (my + x)^2 = a^2 m^2 + b^2 + a^2 + b^2 m^2$$

$$\Rightarrow \quad (x^2 + y^2)(1 + m^2) = (a^2 + b^2)(1 + m^2)$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

which is a circle

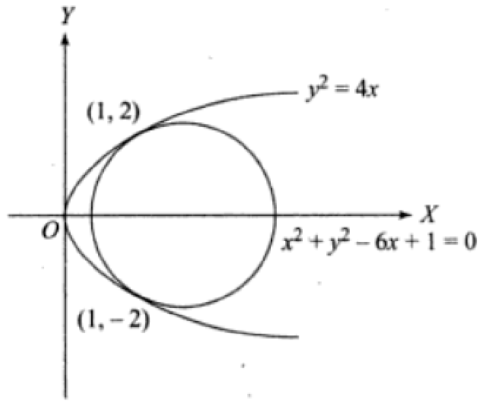
Question

Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$

**Statement-1:** $C_1$ and $C_2$ touch each other exactly at two points.

**Statement-2:** Equation of the tangent at (1, 2) to $C_1$ and $C_2$ both is $x - y + 1 = 0$ and at (1,– 2) is $x + y + 1 = 0$

*Ans.* (*a*)




Solution Solving for the points of intersection we have

$$x^2 + 4x - 6x + 1 = 0$$

$$\Rightarrow \quad (x-1)^2 = 0$$

$$\Rightarrow \quad x = 1 \Rightarrow y = \pm\ 2$$

Thus the two curves meet at (1, 2) and (1, –2)

Tangent at (1, 2) to $y^2 = 4x$ is

$$y(2) = 2(x+1) \Rightarrow x - y + 1 = 0$$

Tangent at (1, 2) to the circle $C_2$ is

$$2x + 1y - 3(x+1) + 1 = 0$$

or $x - y + 1 = 0$ same as the tangent to the curve $C_1$. Similarly the tangent at the point (1, – 2) to the two curves is $x + y + 1 = 0 \Rightarrow$ statement-2 is True and hence statement-1 is also true.

Question

If $b$ and $c$ are the lengths of the segments of any focal chord of a parabola $y^2 = 4ax$, then the length of the semi-latus rectum is

(a) $\dfrac{bc}{b+c}$ (b) $\sqrt{bc}$

(c) $\dfrac{b+c}{2}$ (d) $\dfrac{2bc}{b+c}$

Solution

**(d).** Since the semi latus rectum of a parabola is the harmonic mean between the segments of any focal chord of the parabola.

$\therefore$ $l$ is the harmonic mean between $b$ and $c$.

Hence, $l = \dfrac{2bc}{b+c}$.

Review with more Questions and Solutions

Question

**The eccentricity of the conic $9x^2 + 25y^2 - 18x - 100y - 116 = 0$ is**

**a)** $\frac{5}{4}$  **b)** $\frac{4}{5}$

**c)** $\frac{3}{5}$  **d)** None

**Ans (b)**

The equation can be written as

$9x^2 - 18x - 25y^2 - 100y = 116$

$9(x^2 - 2x) + 25(y^2 - 4y) = 116$

$9(x^2 - 2x + 1) + 25(y^2 - 4y + 4) = 116 + 9 + 100$

$9(x-1)^2 + 25(y-2)^2 = 225$

$$\Rightarrow \frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$$

which is the ellipse with centre at (1, 2)

$a^2 = 25,\ b^2 = 9$

thus

$b^2 = a^2(1 - e^2)$

$\Rightarrow 9 = 25(1 - e^2)$

$$\Rightarrow e = \frac{4}{5}$$

Question

The mirror image of the directrix of the parabola $y^2 = 4(x + 1)$ in the line mirror $x + 2y = 3$ is

(a) $x = -2$ (b) $4y - 3x = 16$

(c) $3x - 4y + 16 = 0$ (d) none of these

Solution

**(c).** Directirx of $y^2 = 4(x + 1)$ is $x = -2$

Any point on $x = -2$ is $(-2, k)$

Now, mirror image $(x, y)$ of $(-2, k)$ in the line $x + 2y = 3$ is given by

$$\frac{x+2}{1} = \frac{y-k}{2} = -2\left(\frac{-2+2k-3}{5}\right)$$

$$\Rightarrow \quad x = \frac{10-4k}{5} - 2 \Rightarrow x = -\frac{4k}{5} \quad \text{...(1)}$$

$$\text{Also, } y = \frac{20-8k}{5} \quad \text{...(2)}$$

From (1) and (2)

$$y = 4 + \frac{3}{5}\left(\frac{5x}{4}\right)$$

or $4y = 16 + 3x$ is the equation of the mirror image of the directrix.

Question

**The parabola whose focus is (- 3, 2) and the directrix is x + y = 4 is**

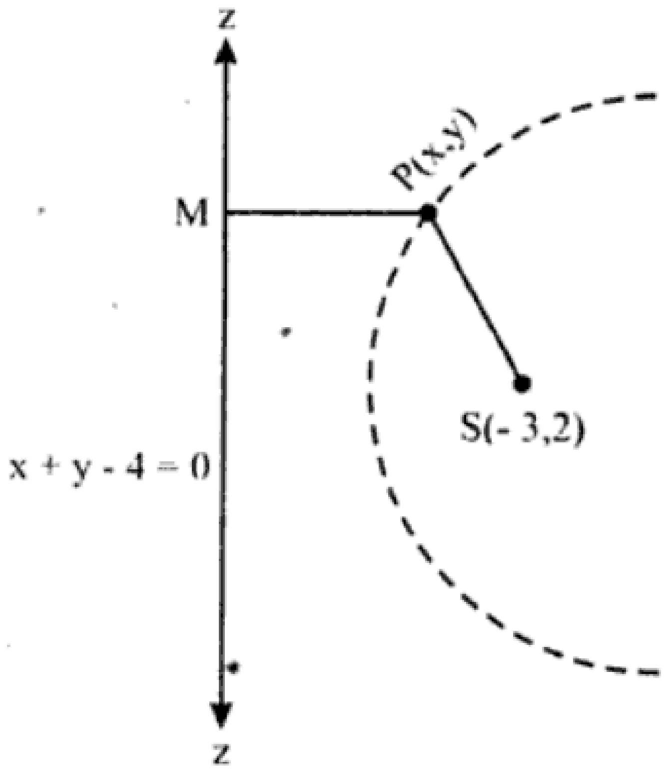
**a)** $y^2 = 8x$

**b)** $y^2 = 8x + 2 + 2y$

**c)** $x^2 + y^2 - 2xy + 20x + 10 = 0$

**d)** $x^2 + 2x = 8y$

**Ans (c)**




Let P(x, y) be the any point on the parabola

We have SP = PM

$\Rightarrow SP^2 = PM^2$

$$\Rightarrow (x + 3)^2 + (y - 2)^2 = \left(\frac{x+y-4}{\sqrt{1+1}}\right)^2$$

$$\Rightarrow 2\left[x^2 + y^2 + 6x - 4y + 13\right]$$
$$= \left[x^2 + y^2 + 16 + 2xy - 8y - 8x\right]$$
$$\Rightarrow x^2 + y^2 - 2xy + 20x + 10 = 0$$

Question

Through the vertex $O$ of a parabola $y^2 = 4x$, chords $OP$ and $OQ$ are drawn at right angles to one another. The locus of the middle point of $PQ$ is

(a) $y^2 = 2x + 8$ (b) $y^2 = x + 8$

(c) $y^2 = 2x - 8$ (d) none of these

Solution

**(c).** Given parabola is $y^2 = 4x$ ...(1)

Here $4a = 4$, $\therefore$ $a = 1$.

Let $P \equiv (t_1^2, 2t_1)$ and $Q \equiv (t_2^2, 2t_2)$.

Slope of $OP = \dfrac{2t_1}{t_1^2} = \dfrac{2}{t_1}$ and slope of $OQ = \dfrac{2}{t_2}$.

Since $OP \perp OQ$, $\therefore$ $\dfrac{4}{t_1 t_2} = -1$ or $t_1 t_2 = -4$ ...(2)

Let $R(\alpha, \beta)$ be the middle point of $PQ$, then

$$\alpha = \frac{t_1^2 + t_2^2}{2} \text{ ...(3) and } \beta = t_1 + t_2 \quad \text{...(4)}$$

From (4), $\beta^2 = t_1^2 + t_2^2 + 2t_1t_2 = 2\alpha - 8$

[From (2) and (3)]

Hence locus of $R(\alpha, \beta)$ is $\mathbf{y^2 = 2x - 8}$.

Question

**The equation of directrix of the parabola $4y^2 + 12x - 12y + 39 = 0$ is**

**a)** $x + \frac{5}{2} = 0$ **b)** $x + \frac{7}{2} = 0$

**c)** $y - \frac{3}{2} = 0$ **d)** None

**Ans (d)**

The equation of the parabola can be written

as $4\left(y^2 - 3y + \frac{9}{4}\right) = -12x - 39 + 9$

$\Rightarrow 4\left(y - \frac{3}{2}\right)^2 = -12\left(x + \frac{5}{2}\right)$

$\Rightarrow y^2 = -4ax,$

Where $x = x + \frac{5}{2}$, $y = y - \frac{3}{2}$ and $a = \frac{3}{4}$

thus the vertex is $\left(-\frac{5}{2}, \frac{3}{2}\right)$

thus equation of directrix is $x = a$

$x + \frac{5}{2} = \frac{3}{4} \Rightarrow x = \frac{-7}{4}$

Question

If the parabolas $y^2 = 4a(x - c_1)$ and $x^2 = 4a(y - c_2)$ touch each other, then the locus of their point of contact is

(a) $xy = 4a^2$ (b) $xy = 2a^2$

(c) $xy = a^2$ (d) none of these

Solution

**(a).** Let $P(x, y)$ be the point of contact.

$\therefore \quad 2y\frac{dy}{dx} = 4a$ and $2x = 4a\frac{dy}{dx}$

$\Rightarrow \quad \frac{4a}{2y} = \frac{2x}{4a} \quad \Rightarrow \quad xy = 4a^2,$

which is the required locus.

Question

**If 2, 5, 9 are the ordinates of vertices of the triangle inscribed in a parabola $2y^2 = x$, then the area of triangle is**

**a)** 42 **b)** 8

**c)** 84 **d)** 72

**Ans (c)**

**Note:**

If $y_1, y_2, y_3$ are the ordinates of vertices of the triangle inscribed in a parabola $y^2 = 4ax$, the area of the triangle is

$$\frac{1}{8a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|$$

Here $a = \frac{1}{8}, y_1 = 2, y_2 = 5, y_3 = 9$

$$\therefore \text{Area} = \frac{1}{8\left(\frac{1}{8}\right)}\left|(2-5)(5-9)(9-2)\right|$$

$=3.4.7$

$=84$ sq.units

Question

If $x_1, x_2, x_3$ as well as $y_1, y_2, y_3$ are in G.P. with the same common ratio, then the points $(x_1, y_1)$, $(x_2, y_2)$ $(x_3, y_3)$

(a) lie on a straight line (b) lie on an ellipse

(c) lie on a circle (d) are vertices of a triangle

Solution

**(a).** Let $\frac{x_2}{x_1} = \frac{x_3}{x_2} = r$ and $\frac{y_2}{y_1} = \frac{y_3}{y_2} = r$

$\Rightarrow \quad x_2 = x_1 r, x_3 = x_1 r^2, y_2 = y_1 r$ and $y_3 = y_1 r^2$.

We have,

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1-r \\ 0 & 0 & 1-r \end{vmatrix}$$

[Applying $R_3 \to R_3 - rR_2$ and $R_2 \to R_2 - rR_1$]

$= 0 \qquad (\because R_2$ and $R_3$ are identical)

Thus, $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ lie on a **straight line.**

Question

Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ may be equal to

(a) 3 (b) 5

(c) 4 (d) none of these

Solution

**(b).** Equations of normals to $y^2 = 4ax$ and $x^2 = 4by$ are given by

$$y = mx - 2am - am^3 \text{ and } y = mx + 2b + \frac{b}{m^2}.$$

For common normals, $2b + \frac{b}{m^2} + 2am + am^3 = 0$

$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$

So, a maximum of 5 normals are possible.

Question

**A point moves so that its distance from (3, 0) is twice the distance from (- 3, 0), then the locus of the point**

**a)** is a circle with centre ( - 5, 1)

**b)** is a straight line

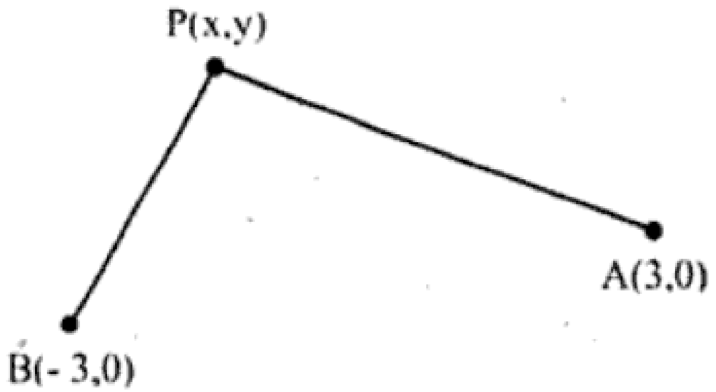
**c)** is an ellipse

**d)** None of the above

Solution

**Ans (d)**

Let the moving point be P(x, y)




Given PA = 2PB

thus $PA^2 = 4PB^2$

$(x-3)^2 + y^2 = 4\ ((x+3)^2 + y^2)$

$x^2+y^2 -6x+9=4x^2 + 4y^2 + 24x + 36$

$3x^2 + 3y^2 + 30x + 27 = 0$

$x^2 + y^2 + 10x + 9=0$

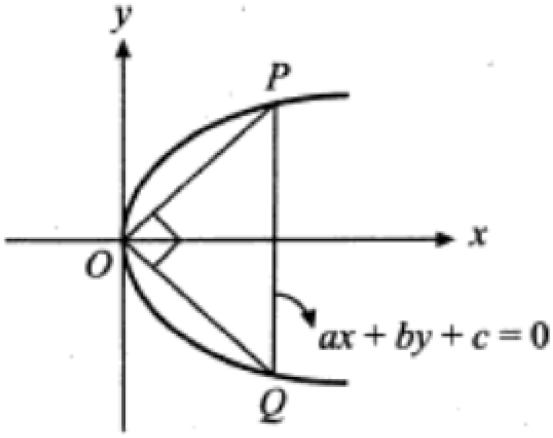
Question

If the segment intercepted by the parabola $y^2 = 4\alpha x$ on the line $ax + by + c = 0$ subtends a right angle at the vertex, then

(a) $4a\alpha + c = 0$
(b) $4b\alpha + c = 0$
(c) $4a\alpha + b = 0$
(d) none of these

Solution

**(a).** Making the equation of parabola $y^2 = 4\alpha x$ homogeneous using the equation of line $ax + by + c = 0$, we get




$$y^2 = 4\alpha x\left(\frac{ax+by}{-c}\right)$$

$\Rightarrow \quad 4a\alpha x^2 + 4b\alpha xy + cy^2 = 0,$

which represents the combined equation of $OP$ and $OQ$.

Since $\angle POQ = 90°$, coefficient of $x^2$ + coefficient of $y^2$ $= 0$

$\Rightarrow \quad 4a\alpha + c = 0$

Question

Let $ax + by + c = 0$ be a variable straight line, where $a$, $b$ and $c$ are first, third and seventh terms of an increasing A.P. Then, the variable straight line always passes through a fixed point which lies on

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 13$
(c) $y^2 = 2x$ (d) $2x + 3y = 9$

Solution

**(b).** Let $d$ be the common difference of A.P., then

$b = a + 2d$ and $c = a + 6d$. Clearly, $(b - a) \times 3 = c - a$

$\Rightarrow \quad 2a - 3b + c = 0$

Thus, the straight line $ax + by + c = 0$ passes through the point $(2, -3)$ which also satisfies $x^2 + y^2 = 13$

Question

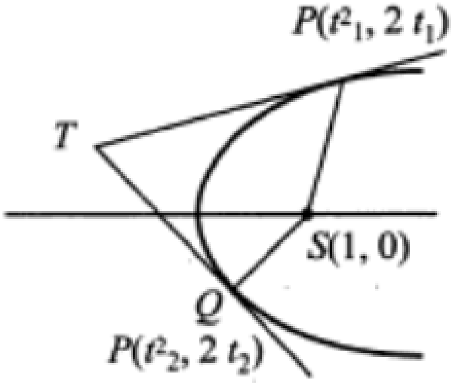
The tangents at two points $P$ and $Q$ on the parabola $y^2 = 4x$ intersect at $T$. If $SP$, $ST$ and $SQ$ are equal to $a$, $b$ an $c$ respectively, where $S$ is the focus, then the roots of the equation $ax^2 + 2bx + c = 0$ are

(a) real and equal (b) real and unequal

(c) complex numbers (d) irrational

Solution

**(a).** The tangents at the points $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ intersect at the point $T(t_1t_2, t_1 + t_2)$.




Now, $a = SP = 1 + t_1^2$ and $c = SQ = 1 + t_2^2$

$\therefore \quad b^2 = ST^2 = (t_1t_2 - 1)^2 + (t_1 + t_2)^2$

$= t_1^2 + t_2^2 + 1 + t_1^2t_2^2$

$= (1 + t_1^2)(1 + t_2^2) = ac$

$\therefore$ Roots of the equation $ax^2 + 2bx + c = 0$ are real and equal.

Question

**The equation of the circle which has the centres of the circles whose equations are $x^2 + y^2 - 16x - 18y + 20 = 0$ and $x^2 + y^2 - 3x + y - 4 = 0$ as the end point of its diameter is**

**a)** $x^2 + y^2 - 17x - 16y = 0$

**b)** $x^2 + y^2 - 16x + 17y + 15 = 0$

**c)** $x^2 + y^2 + 16x - 17y + 15 = 0$

**d)** None

Solution

**Ans (a)**

The centres of the given circles are $C_1(8,9)$ and $C_2\left(\frac{3}{2},-\frac{-1}{2}\right)$ $C_1$, $C_2$ as the end points of diameter is

$$\left(x-\frac{3}{2}\right)(x-8)+\left(y+\frac{1}{2}\right)(y-9)=0$$

$$(2x-3)(x-8)+(2y+1)(y-9)=0$$

Question

The centroid of the triangle formed by the feet of the normals from the point $(h, k)$ to the parabola $y^2 + 4ax = 0$, $(a > 0)$ lies on

(a) $x$-axis (b) $y$-axis

(c) $x = h$ (b) $y = k$

Solution

**(a).** Co-ordinates of any point on the parabola $y^2 = -4ax$ are $(-at^2, 2at)$.

Equation of the normal at $(-at^2, 2at)$ is

$$y - xt = 2at + at^3$$

If the normal passes through the point $(h, k)$, then

$$k - th = 2at + at^3$$

or $\quad at^3 + (2a + h)t - k = 0,$

which is a cubic equation whose three roots $t_1$, $t_2$, $t_3$ are the parameters of the feet of the three normals.

$$\therefore \quad \text{Sum of the roots} = t_1 + t_2 + t_3 = -\frac{\text{Coefficient of } t^2}{\text{Coefficient of } t^3} = 0$$

$\therefore$ Centroid of the triangle formed by the feet of the normals

$$= \left(-\frac{a}{3}(t_1^2 + t_2^2 + t_3^2), \frac{2a}{3}(t_1 + t_2 + t_3)\right)$$

$$= \left(-\frac{a}{3}(t_1^2 + t_2^2 + t_3^2), 0\right)$$

which, clearly, lies on the $x$-axis.

Question

**The length of the intercept made by the line y = 2x + 1 on the circle $x^2 + y^2 = 2$ is**

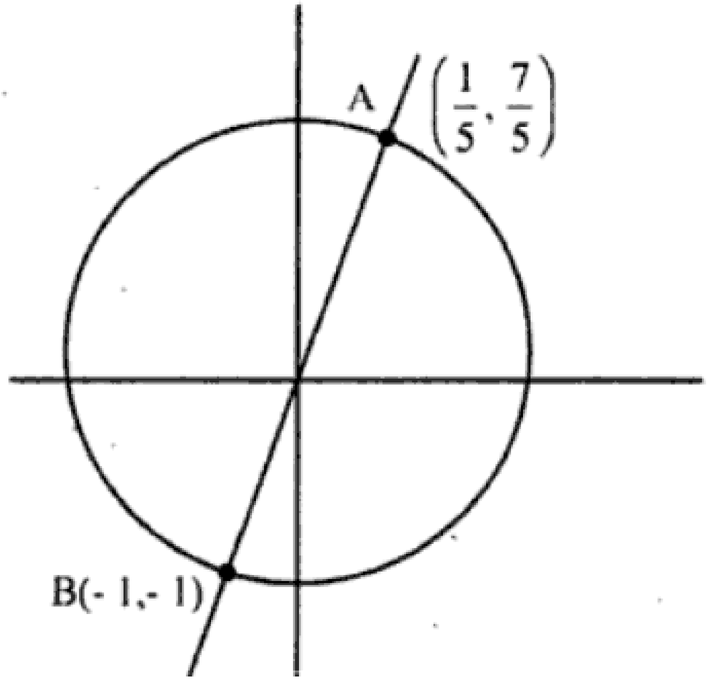
a) $\frac{6}{\sqrt{5}}$ b) $6\sqrt{5}$

c) $6\sqrt{2}$ d) None

Solution

**Ans (a)**

Solving the equations $y=2x+1$ and $x^2 + y^2 = 2$, we get $x^2+(2x+1)^2 = 2$



$\Rightarrow 5x^2 + 4x - 1 = 0$

$\Rightarrow x = \frac{1}{5}, -1$

$\Rightarrow y = \frac{7}{5}, -1$

**Hence the coordinates of the points of intersection are**

$$A\left(\frac{1}{5}, \frac{7}{5}\right) \text{ and } B(-1, -1)$$

$$AB = \sqrt{\left(\frac{1}{5}+1\right)^2 + \left(\frac{7}{5}+1\right)^2} = \sqrt{\frac{36}{25} + \frac{144}{25}}$$

$$= \frac{\sqrt{180}}{5} = \frac{6}{\sqrt{5}}$$

Question

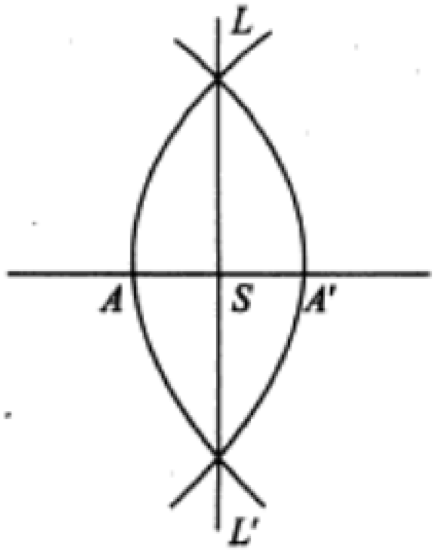
Given the two ends of the latus rectum, the maximum number of parabolas that can be drawn is

(a) 1 (b) 2

(c) 3 (d) none of these

Solution

**(b).** $L$ and $L'$ are the ends of latus rectum. $S$ bisects $LL'$. As $A'$ is perpendicular bisector of $LL'$, where $AS = \frac{1}{4}LL' = A'S$.




Clearly, two parabolas are possible.

Question

The angle formed by the abscissa and the tangent to the parabola $y = x^2 + 4x - 17$ at the point $\left(\frac{5}{2}, -\frac{3}{4}\right)$ is

(a) $\tan^{-1} 2$ (b) $\tan^{-1} 5$

(c) $\tan^{-1} 7$ (d) None of these.

Solution

(*d*) Slope of $x$-axis is 0.

$y = x^2 + 4x - 17 \Rightarrow \frac{dy}{dx} = 2x + 4$

$\therefore$ slope of tangent to parabola at $P\left(\frac{5}{2}, -\frac{3}{4}\right)$

$$= 2\left(\frac{5}{2}\right) + 4 = 9$$

If $\theta$ is the angle between $x$-axis and the tangent at P then $\tan\theta = 9 \Rightarrow \theta = \tan^{-1} 9$.

Question

A line $L$ passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola in two distinct points. If $m$ be the slope of the line $L$, then

(a) $m \in R - \{0\}$ (b) $-1 < m < 1$

(c) $m < -1$ or $m > 1$ (d) none of these

Solution

**(a).** The focus of the parabola $y^2 = 4(x - 1)$ is (2, 0). Any line through the focus is

$$(y - 0) = m(x - 2), \text{ i.e. } y = m(x - 2).$$

It will meet the given parabola if

$m^2(x - 2)^2 = 4(x - 1)$

or $m^2x^2 - 4(m^2 + 1)x + 4(m^2 + 1) = 0$

If $m \neq 0$, discriminant $= 16(m^2 + 1)^2 - 16m^2(m^2 + 1) = 0$

$= 16(m^2 + 1) > 0$ for all $m$

But if $m = 0$, then $x$ does not have two real distinct values

$\therefore \quad m \in R - \{0\}$

Question

**The parametric equations of the circle $x^2 + y^2 + 8x - 6y = 0$ are**

a) $x = 4 + 5\cos\theta,\ y = 3 + 5\sin\theta$

b) $x = -4 + 5\cos\theta,\ y = 3 + 5\sin\theta$

c) $x = 4 + 5\cos\theta,\ y = -3 + 5\sin\theta$

d) $x = -4 + 5\cos\theta,\ y = -3 + 5\sin\theta$

Solution

**Ans (b)**

The circle is $(x + 4)^2 + (y - 3)^2 = 25$

thus the parametric equation is

$x + 4 = 5\cos\theta,\ y - 3 = 5\sin\theta$

ie., $x = -4 + 5\cos\theta,\ y = 3 + 5\sin\theta$

Question

If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance $b$ from the vertex is $c$ , then

(a) $a^2c = 4b^3$

(b) $b^2c = 4a^3$

(c) $c^2b = 4a^3$

(d) none of these

Solution

**(b).** Let the ends of the focal chord be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. Then $t_1t_2 = -1$.

Equation of the focal chord is

$$(t_1 + t_2)y = 2x + 2at_1t_2$$

**Given:** $b = \dfrac{2at_1t_2}{\sqrt{(t_1+t_2)^2+4}} = \dfrac{-2a}{\sqrt{2+t_1^2+t_2^2}}$

Also, $c^2 = a^2(t_1^2 - t_2^2)^2 + 4a^2(t_1 - t_2)^2$

$= a^2(t_1 - t_2)^2[(t_1+t_2)^2 + 4]$

$= a^2(t_1^2 + t_2^2 + 2)^2 \qquad [\because\ t_1t_2 = -1]$

$\therefore \qquad c = a(t_1^2 + t_2^2 + 2)$

Now, $b^2 = \dfrac{4a^2}{t_1^2 + t_2^2 + 2} = \dfrac{4a^2}{c/a} = \dfrac{4a^3}{c}$

$\therefore \qquad b^2c = 4a^3.$

Question

**The angle between the tangents drawn from (0, 0) to the circle $x^2 + y^2 + 4x - 6y + 4 = 0$ is**

**a)** $\sin^{-1}\left(\dfrac{5}{13}\right)$ **b)** $\sin^{-1}\left(\dfrac{5}{12}\right)$

**c)** $\sin^{-1}\left(\dfrac{12}{13}\right)$ **d)** $\dfrac{\pi}{2}$

Solution

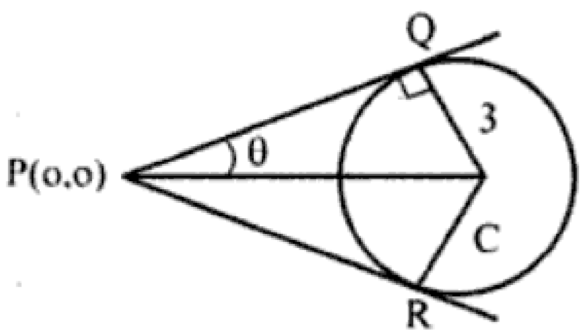
**Ans (c)**

The centre of the circle, c = (- 2, 3)

radius of the circle, $r = \sqrt{4+9-4} = 3$

PQ = length of the tangent from P(0, 0) to the circle

$= \sqrt{4} = 2.$




From ΔPQC, we have $\tan\theta = \frac{QC}{PQ} = \frac{3}{2}$

$$\therefore \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\left(\frac{3}{2}\right)}{1+\frac{9}{4}} = \frac{12}{13}$$

thus $2\theta = \sin^{-1}\left(\frac{12}{13}\right)$.

Question

If from a point, the two tangents drawn to the parabola $y^2 = 4ax$ are normals to the parabola $x^2 = 4by$, then

(a) $a^2 > 8b^2$ (b) $b^2 > 8a^2$

(c) $a^2 < 8b^2$ (d) none of these

Solution

**(a).** The coordinates of any point on the parabola $x^2 = 4by$ are $(2bt, bt^2)$.

For the parabola $x^2 = 4by$, $\dfrac{dy}{dx} = \dfrac{x}{2b}$.

Slope of the normal at $(2bt, bt^2) = -\dfrac{2b}{2bt} = -\dfrac{1}{t}$

$\therefore$ Equation of normal is $y - bt^2 = -\dfrac{1}{t}(x - 2bt)$

or $y = -\dfrac{x}{t} + 2b + bt^2$

It will touch the parabola $y^2 = 4ax$ if

$$2b + bt^2 = \frac{a}{-1/t} \qquad \left(\because c = \frac{a}{m}\right)$$

$\Rightarrow$ $bt^2 + at + 2b = 0$

For distinct real roots, discriminant > 0

$\Rightarrow$ $a^2 - 8b^2 = 0$ or $a^2 > 8b^2$

Question

The distance between the point (1, 1) and the tangent to the curve $y = e^{2x} + x^2$ drawn from the point $x = 0$ is

(a) $\dfrac{1}{\sqrt{5}}$ (b) $\dfrac{-1}{\sqrt{5}}$

(c) $\dfrac{2}{\sqrt{5}}$ (d) $\dfrac{-2}{\sqrt{5}}$.

Solution

**(c) Putting $x = 0$ in $y = e^{2x} + x^2$ ...(1)**

**we get $y = 1$**

**$\therefore$ the given point is P(0, 1)**

**From (1),** $\frac{dy}{dx} = 2e^{2x} + 2x$

$\Rightarrow \left[\frac{dy}{dx}\right]_P = 2$

**$\therefore$ equation of tangent at P to (1) is**

$$y - 1 = 2(x - 0) \Rightarrow 2x - y + 1 = 0 \quad ...(2)$$

**$\therefore$ Required distance**

= Length of $\perp$ from (1, 1) to (2)

$$= \frac{2 - 1 + 1}{\sqrt{4 + 1}} = \frac{2}{\sqrt{5}}.$$

Question

If two tangents drawn from the point $(x_1, y_1)$ to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other, then

(a) $2y_1^2 = 9x_1$ (b) $2x_1^2 = 9y_1$

(c) $4y_1^2 = 9x_1$ (d) none of these

Solution

**(a).** The equation of any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

If it passes through the point $(x_1, y_1)$, then

$$y_1 = mx_1 + \frac{1}{m} \quad \text{or} \quad x_1 m^2 - y_1 m + 1 = 0$$

Its roots are given to be $m_1$ and $2m_1$

$$\therefore \quad m_1 + 2m_1 = \frac{y_1}{x_1} \Rightarrow 3m_1 = \frac{y_1}{x_1}$$

and $$m_1 \cdot 2m_1 = \frac{1}{x_1} \Rightarrow 2m_1^2 = \frac{1}{x_1}$$

$$\therefore \quad 2\left(\frac{y_1}{3x_1}\right)^2 = \frac{1}{x_1} \quad \text{or} \quad 2y_1^2 = 9x_1$$

Question

**The circles $x^2 + y^2 - 8x + 6y + 21 = 0$, $x^2 + y^2 + 4x - 10y - 115 = 0$**

**a)** touch externally

**b)** touch internally

**c)** intersect at two points

**d)** None

Solution

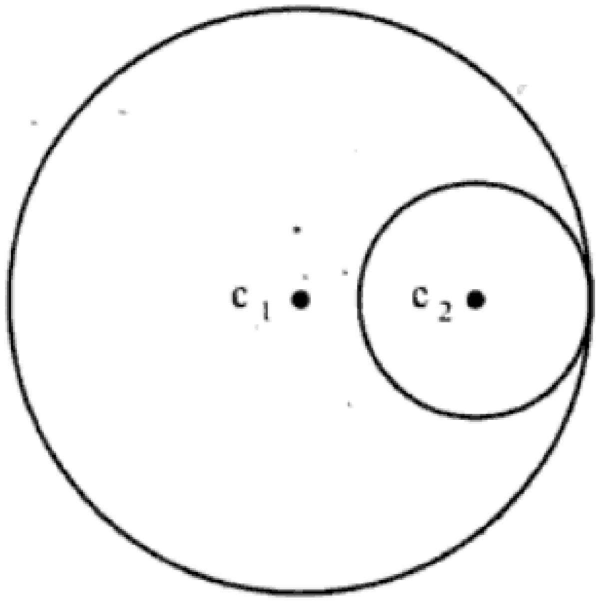
**Ans (b)**

the centres of the circles are $C_1 = (4, -3)$, $C_2 = (-2, 5)$ the radii are

$r_1 = \sqrt{16+9-21} = 2$, $r_2 = \sqrt{4+25+115} = 12$

Here $C_1C_2 = \sqrt{36+64} = 10$

Since $C_1C_2 = |r_1 - r_2|$, the circles touch each other internally.




Question

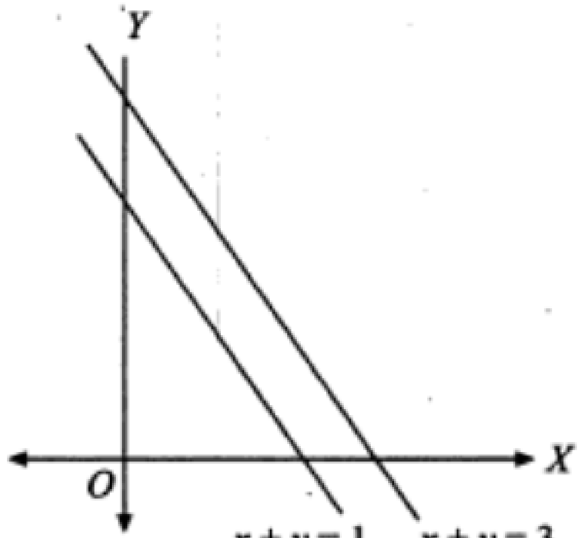
If the focus of the parabola $(y-\beta)^2 = 4(x-\alpha)$ always lies between the lines $x + y = 1$ and $x + y = 3$, then

(a) $1 < \alpha + \beta < 2$ (b) $0 < \alpha + \beta < 1$

(c) $0 < \alpha + \beta < 2$ (d) none of these

Solution

**(c).** The coordinates of the focus of the given parabola are $(\alpha + 1, \beta)$.




Clearly, focus must lie to the opposite side of the origin w.r.t. the line $x + y - 1 = 0$ and same side as origin with respect to the line $x + y - 3 = 0$. Hence, $\alpha + \beta > 0$ and $\alpha + \beta < 2$.

Question

At (0, 0), the curve $y^2 = x^3 + x^2$

(*a*) touches $x$-axis

(*b*) bisects the angle between the axes

(*c*) makes an angle of 60° with $ox$

(*d*) None of these.

Solution

(b) $y^2 = x^3 + x^2 \Rightarrow 2y\frac{dy}{dx} = 3x^2 + 2x$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2+2x}{2y} = \frac{3x^2+2x}{2\sqrt{x^3+x^2}} = \frac{3x+2}{2\sqrt{1+x}}$

$\therefore \left.\frac{dy}{dx}\right|_{(0,0)} = \frac{2}{2} = 1 \Rightarrow \theta = 45°$

$\therefore$ the curve bisects the angle between the axes.

Question

The tangent to the curve $y = 2x^2 - x + 1$ is parallel to the line $y = 3x + 9$ at the point

(a) (2, 3) (b) (2, – 1)

(c) (2, 1) (d) (1, 2).

Solution

(d) $y = 2x^2 - x + 1$

$\Rightarrow \frac{dy}{dx} = 4x - 1$

Also, slope of $y = 3x + 9$ is 3.

$\therefore 4x - 1 = 3 \Rightarrow x = 1$

$\therefore$ From (1), $y = 2(1)^2 - 1 + 1 = 2$

$\therefore$ Point is (1, 2).

Question

The number of common tangents to the circles $x^2 + y^2 - 2x + 4y + 4 = 0$, $x^2 + y^2 + 4x - 2y + 1 = 0$ are

a) 0 b) 1

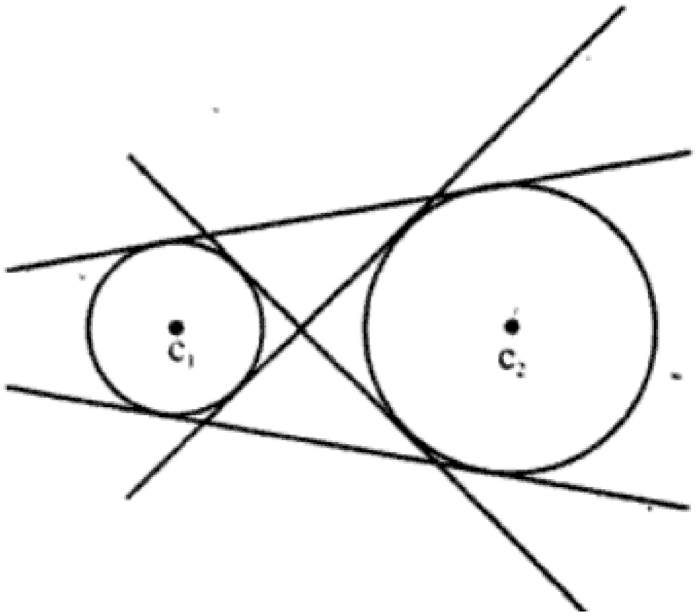
c) 2 d) 4

Solution

**Ans (d)**

The centres of the circles are $c_1 = (1, -2)$, $c_2 = (-2, 1)$ the radii are

$r_1 = \sqrt{1+4-4} = 1$, $r_2 = \sqrt{4+1-1} = 2$.




Here $C_1C_2 = \sqrt{9+9} = 3\sqrt{2}$.

Since $C_1C_2 > r_1 + r_2$, the circles are non-overlapping circles thus 4 common tangents.

Question

**The radius of the director circle of the ellipse**

$\frac{x^2}{6} + \frac{y^2}{4} = 1$ **is**

**a)** $\sqrt{10}$ **b)** 10

**c)** 5 **d)** $\sqrt{5}$

Solution

**Ans (a)**

**Note:**

The locus of point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$ called director circle of the ellipse.

$\therefore x^2 + y^2 = 6 + 4$

ie., $x^2 + y^2 = 10$, is the equation of the director circle whose radius is $\sqrt{10}$.

Question

**The locus of the point of intersection of feet of perpendicular from focus on the tangent drawn to the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **(a > b) is** $x^2 + y^2 = 7$**, then**

**a)** $a = 7$ **b)** $b = 7$

**c)** $a^2 = 7$ **d)** $b^2 = 7$

Solution

**Ans (c)**

**Note:**

The locus of the point of intersection of feet of perpendicular from focus on the tangent drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2$ called auxiliary circle.

$\therefore a^2 = 7.$

Question

**The equation of the normal to the ellipse $\frac{x^2}{10}+\frac{y^2}{5}=1$ at $(\sqrt{8}, 1)$ is**

a) $10x + 5y = 1$ b) $y=\sqrt{2}(x+1)$

c) $x=\sqrt{2}(y+1)$ d) $y=\sqrt{8}(x+1)$

Solution

**Ans (c)**

The equation of normal is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

ie., $\frac{10x}{\sqrt{8}}-\frac{5y}{1}=10-5$

$$\frac{2x}{\sqrt{8}}-y=1 \qquad \frac{x}{\sqrt{2}}=1+y$$

$$\Rightarrow x=\sqrt{2}(1+y)$$

Question

If the lines joining the origin to the intersection of the line $y = mx + 2$ and the curve $x^2 + y^2 = 1$ are at right angles, then

(a) $m^2 = 1$ (b) $m^2 = 3$

(c) $m^2 = 7$ (d) $2m^2 = 1$

*Ans.* (c)

**Solution** Joint equation of the lines joining the origin and the point of intersection of the line $y = mx + 2$ and the curve $x^2 + y^2 = 1$ is

$$x^2 + y^2 = \left(\frac{y - mx}{2}\right)^2$$

$$\Rightarrow \quad x^2 (4 - m^2) + 2mxy + 3y^2 = 0$$

Since these lines are at right angles

$$4 - m^2 + 3 = 0 \Rightarrow m^2 = 7.$$

Question

**The equations of the tangents to the ellipse**

**$\frac{x^2}{28} + \frac{y^2}{16} = 1$ which makes an angle $60^\circ$ with**

**the major axis are**

**a)** $y = \sqrt{3}x \pm 10$ **b)** $y = \sqrt{3}x \pm \sqrt{65}$

**c)** $x = \sqrt{3}y \pm 28$ **d)** $x = \sqrt{3}y \pm 7$

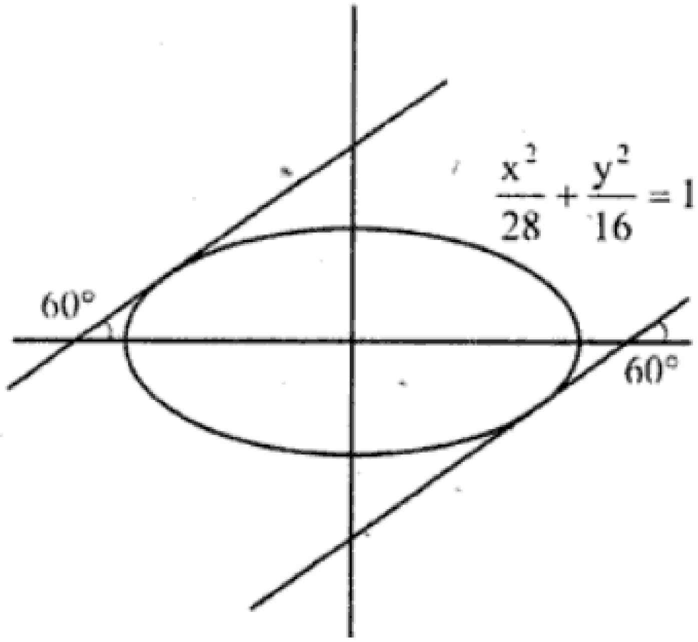
Solution

**Ans (a)**

Here slope of tangent = tan 60°

$m = \sqrt{3}$

∴ The equation of tangent is




$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$y = \sqrt{3}x \pm \sqrt{28 \times 3 + 16}$$

$$y = \sqrt{3}x \pm 10.$$

Question

**The number of tangents to $\frac{x^2}{25}+\frac{y^2}{16}=1$ through (5, 0) is**

**a)** 0 **b)** 1

**c)** 2 **d)** 3

Solution

**Ans (b)**

Since the points (5, 0) lies on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ there is only one tangent (5, 0)

Question

**The tangents at any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the tangents at the vertices A and $A^1$ in L and M respectively. then AL. $A^1M$ =**

**a)** $a^2$ **b)** $b^2$

**c)** ab **d)** $a^2b^2$

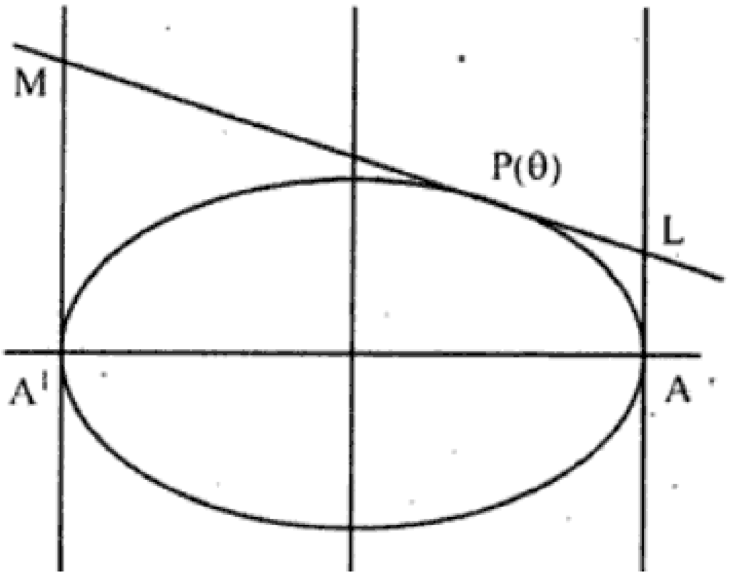
Solution

**Ans (b)**

The equation of tangent at P(θ) to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad ...(1)$$



at L, x = a $\therefore \frac{a\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$\Rightarrow y = \frac{b}{\sin\theta}(1-\cos\theta)$

$\Rightarrow AL = \frac{b}{\sin\theta}(1-\cos\theta)$

at , x = - a $\Rightarrow y = \frac{b}{\sin\theta}(1+\cos\theta)$

$\Rightarrow A'M = \frac{b}{\sin\theta}(1+\cos\theta)$

thus $AL.\ A'M = \frac{b^2}{\sin^2\theta}(1-\cos^2\theta) = b^2.$

Question

If $a$, $b$, $c$ form a G.P., then twice the sum of the ordinates of the points of intersection of the line $ax + by + c = 0$ and the curve $x + 2y^2 = 0$ is

(a) $\frac{b}{a}$ (b) $\frac{c}{a}$

(c) $\frac{a}{c}$ (d) $\frac{a}{b}$

Solution

**(a).** Let $a$, $b$, $c$ be in G.P. with common ratio $r$.

Then, $b = ar$ and $c = ar^2$.

So, the equation of the line is $ax + by + c = 0$

$\Rightarrow \quad ax + ary + ar^2 = 0 \quad \Rightarrow \quad x + ry + r^2 = 0$

This line cuts the curve $x + 2y^2 = 0$

Eliminating $x$, we get $2y^2 - ry + r^2 = 0$

If the roots of the quadratic equation are $y_1$ and $y_2$, then

$$y_1 + y_2 = \frac{r}{2} \Rightarrow 2(y_1 + y_2) = r = \frac{b}{a} = \frac{c}{b}.$$

Question

If $a, b, c$ are in A.P., $a, x, b$ are in G.P. and $b, y, c$ are in G.P., the point $(x, y)$ lies on

(a) a straight line
(b) a circle
(c) an ellipse
(d) a hyperbola

*Ans.* (b)

**Solution** We have $2b = a + c$, $x^2 = ab$, $y^2 = bc$ so that $x^2 + y^2 = b(a + c) = 2b^2$ which is a circle.

Question

**The second degree equation $x^2 + 3xy + 2y^2 + 3x + 5y + 2 = 0$ represents**

**a)** parabola
**b)** ellipse
**c)** hyperbola
**d)** pair of straight lines

Solution

**Ans (d)**

Here a=1, $h = \frac{3}{2}$, b=2, $g = \frac{3}{2}$, $f = \frac{5}{2}$, c=2

thus abc+ 2fgh-af²-bg²-ch²

$$= 1.(2)(2)+2\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)$$

$$-1\left(\frac{5}{2}\right)^2 -2\left(\frac{3}{2}\right)^2 -2\left(\frac{3}{2}\right)^2 = 0$$

**thus the second dgree equation represents pair of straight lines.**

:-{D

To recall standard integrals

| $f(x)$ | $\int f(x)dx$ | $f(x)$ | $\int f(x)dx$ |
|---|---|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1}$ $(n \neq -1)$ | $[g(x)]^n g'(x)$ | $\frac{[g(x)]^{n+1}}{n+1}$ $(n \neq -1)$ |
| $\frac{1}{x}$ | $\ln\|x\|$ | $\frac{g'(x)}{g(x)}$ | $\ln\|g(x)\|$ |
| $e^x$ | $e^x$ | $a^x$ | $\frac{a^x}{\ln a}$ $(a > 0)$ |
| $\sin x$ | $-\cos x$ | $\sinh x$ | $\cosh x$ |
| $\cos x$ | $\sin x$ | $\cosh x$ | $\sinh x$ |
| $\tan x$ | $-\ln\|\cos x\|$ | $\tanh x$ | $\ln\cosh x$ |
| $\operatorname{cosec} x$ | $\ln\left\|\tan\frac{x}{2}\right\|$ | $\operatorname{cosech} x$ | $\ln\left\|\tanh\frac{x}{2}\right\|$ |
| $\sec x$ | $\ln\|\sec x + \tan x\|$ | $\operatorname{sech} x$ | $2\tan^{-1} e^x$ |
| $\sec^2 x$ | $\tan x$ | $\operatorname{sech}^2 x$ | $\tanh x$ |
| $\cot x$ | $\ln\|\sin x\|$ | $\coth x$ | $\ln\|\sinh x\|$ |
| $\sin^2 x$ | $\frac{x}{2} - \frac{\sin 2x}{4}$ | $\sinh^2 x$ | $\frac{\sinh 2x}{4} - \frac{x}{2}$ |
| $\cos^2 x$ | $\frac{x}{2} + \frac{\sin 2x}{4}$ | $\cosh^2 x$ | $\frac{\sinh 2x}{4} + \frac{x}{2}$ |

| $f(x)$ | $\int f(x)\,dx$ | $f(x)$ | $\int f(x)\,dx$ |
|---|---|---|---|
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a}\tan^{-1}\frac{x}{a}$ | $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$ |
| | $(a>0)$ | $\frac{1}{x^2-a^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert > a>0)$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\frac{x}{a}$ | $\frac{1}{\sqrt{a^2+x^2}}$ | $\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$ |
| | $(-a<x<a)$ | $\frac{1}{\sqrt{x^2-a^2}}$ | $\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$ |
| $\sqrt{a^2-x^2}$ | $\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$ | $\sqrt{a^2+x^2}$ | $\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$ |
| | $\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$ | $\sqrt{x^2-a^2}$ | $\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$ |

Some series Expansions -

$$\frac{\pi}{2} = \left(\frac{2}{1}\frac{2}{3}\right)\left(\frac{4}{3}\frac{4}{5}\right)\left(\frac{6}{5}\frac{6}{7}\right)\left(\frac{8}{7}\frac{8}{9}\right)\ldots$$

$$\pi = \frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\frac{4}{13}-\cdots$$

$$\frac{\pi}{4} = \frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\cdot$$

$$\pi = \sqrt{12}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$$

**Solve a series problem**

If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ upto $\infty = \frac{\pi^2}{6}$, then value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ up to $\infty$ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$

*Ans.* (c)

**Solution** We have $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ upto $\infty$

$$= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\cdots \text{ upto } \infty$$

$$-\frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$= \frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\ldots\infty=\frac{\pi^2}{12}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\infty=\frac{\pi^2}{24}$$

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\frac{x^4}{9!}-\frac{x^5}{11!}+\ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \le x < 1)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} \cdots + \frac{2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots + \frac{2\left(2^{2n-1}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} - \cdots \quad 0 < |x| < \pi$$

$$\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \cdots$$

$$\sec x = 1 + \frac{x^2}{\lfloor 2} + \frac{5\,x^4}{\lfloor 4} + \cdots$$

$$\log\,(\cos x) = -\frac{x^2}{\lfloor 2} - \frac{2\,x^4}{\lfloor 4} - \cdots$$

$$\log\,(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \quad |x| < 1$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$= \frac{\pi}{2} - \left( x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \right) \quad |x| < 1$$

$$\tan^{-1} x = \begin{cases} x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots & |x| < 1 \\ \pm\dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} + \cdots & \begin{cases} + \text{ if } x \geq 1 \\ - \text{ if } x \leq -1 \end{cases} \end{cases}$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$= \frac{\pi}{2} - \left( \frac{1}{x} + \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots \right) \quad |x| > 1$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

$$= \frac{1}{x} + \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots \quad |x| > 1$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$= \begin{cases} \dfrac{\pi}{2} - \left( x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots \right) & |x| < 1 \\ p\pi + \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} + \cdots & \begin{cases} p = 0 \text{ if } x \geq 1 \\ p = 1 \text{ if } x \leq -1 \end{cases} \end{cases}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right]$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1} \quad (x > 0)$$

$$\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x-1}{x}\right)^n \quad (x > \frac{1}{2})$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}(x-1)^n \quad (0 < x \le 2)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}x^n \quad (|x| < 1)$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\infty \, (-1 \le x < 1)$$

$$\log_e(1+x) - \log_e(1-x) =$$

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty\right) (-1 < x < 1)$$

$$\log_e\left(1 + \frac{1}{n}\right) = \log_e \frac{n+1}{n} = 2 \quad \left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\infty\right]$$

$$\log_e\ (1 + x) + \log_e\ (1 - x) = \log_e(1-x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \ldots\infty\right)(-1 < x < 1)$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \quad = \frac{1}{1 \cdot 2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$$

# Important Results

**(i)** (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1+\tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1+\cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x} dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x} dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x + \operatorname{cosec}^n x} dx$ where, $n \in R$

**(ii)** $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \frac{\pi}{4}$

**(iii)** (a) $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$

(b) $\int_0^{\pi/2} \log \tan x \, dx = \int_0^{\pi/2} \log \cot x \, dx = 0$

(c) $\int_0^{\pi/2} \log \sec x \, dx = \int_0^{\pi/2} \log \operatorname{cosec} x \, dx = \frac{\pi}{2} \log 2$

**(iv)** (a) $\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

(b) $\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

(c) $\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^n+1}$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + C$$

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Good Luck to you for your Preparations, References, and Exams

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